THE APRIL MEETING OF THE AMERICAN MATHEMATICAL SOCIETY.

THE one hundred and forty-eighth regular meeting of the Society was held in New York City on Saturday, April 30, 1910, extending through the usual morning and afternoon sessions. The following forty-one members were present:

Mr. E. S. Allen, Mr. F. W. Beal, Professor W. J. Berry, Dr. E. G. Bill, Professor G. D. Birkhoff, Professor E. W. Brown, Mr. R. D. Carmichael, Miss Emily Coddington, Professor F. N. Cole, Professor J. L. Coolidge, Professor L. P. Eisenhart, Professor H. B. Fine, Professor T. S. Fiske, Professor C. C. Grove, Dr. G. F. Gundelfinger, Dr. L. C. Karpinski, Professor Edward Kasner, Mr. W. C. Krathwohl, Professor W. R. Longley, Professor J. H. Maclagan-Wedderburn, Dr. H. F. MacNeish, Dr. Emilie N. Martin, Mr. A. R. Maxson, Mr. H. H. Mitchell, Professor C. L. E. Moore, Professor Frank Morley, Professor G. D. Olds, Professor W. F. Osgood, Dr. H. B. Phillips, Mr. H. W. Reddick, Professor R. G. D. Richardson, Mr. L. P. Siceloff, Professor D. E. Smith, Professor P. F. Smith, Dr. W. M. Strong, Professor J. H. Tanner, Professor C. B. Upton, Professor Oswald Veblen, Mr. H. E. Webb, Miss M. E. Wells, Professor H. S. White.

Ex-President W. F. Osgood occupied the chair at the morning session, Ex-President T. S. Fiske and Professor Frank Morley at the afternoon session. The Council announced the election of the following persons to membership in the Society: Mr. F. W. Beal, Princeton University; Professor W. J. Berry, Brooklyn Polytechnic Institute; Mr. J. K. Lamond, Yale University; Mr. R. M. Mathews, University High School, Chicago, Ill.; Professor F. E. Miller, Otterbein University; Mr. J. E. Rowe, Johns Hopkins University; Mr. W. H. Terrell, Clyde, N. C.; Mr. George Wentworth, Exeter, N. H.; Mr. W. A. Wilson, Yale University. Eight applications for membership in the Society were received.

Professor Bôcher was elected by the Council to succeed Professor Osgood as a member of the Editorial Board of the *Transactions*. Professor Dickson was appointed to fill the unexpired term of Professor E. B. Van Vleck, who retires from the Editorial Board in July.

It was decided to hold the April, 1911, meeting of the Society at Chicago, Ill. The publication of the lectures delivered at the Princeton Colloquium by Professors G. A. Bliss and Edward Kasner was placed in charge of the Committee of Publication. The New Haven Colloquium Lectures have recently been issued by the Yale University Press.

The following papers were read at this meeting:

(1) Dr. H. B. PHILLIPS: "Application of Gibbs's indeterminate product to the algebra of linear systems."

(2) Dr. H. B. PHILLIPS: "Concerning a class of surfaces associated with polygons on a quadric surface."

(3) Professor VIRGIL SNYDER: "Conjugate line congruences contained in a bundle of quadric surfaces."

(4) Professor W. B. CARVER: "Ideals of a quadratic number field in canonic form."

(5) Professor G. A. MILLER: "On a method due to Galois."

(6) Dr. E. H. TAYLOR: "On the transformation of the boundary in conformal mapping."

(7) Professor W. B. FITE: "Concerning the invariant points of commutative collineations."

(8) Professor R. G. D. RICHARDSON: "On the saddle point in the theory of maxima and minima and in the calculus of variations."

(9) Mr. H. H. MITCHELL: "Note concerning the subgroups of the linear fractional group $LF(2, p^n)$."

(10) Mr. H. H. MITCHELL: "The subgroups of the linear group $LF(3, p^n)$."

(11) Professor C. L. E. MOORE: "Some infinitesimal properties of five-parameter families of lines in space of four dimensions."

(12) Professor EDWARD KASNER: "Forces depending on the time, and a related transformation group."

(13) Professor F. H. SAFFORD: "Sturm's method of integrating $dx/\sqrt{X} + dy/\sqrt{Y} = 0$."

(14) Dr. G. F. GUNDELFINGER: "On the geometry of line elements in the plane with reference to osculating circles."

In the absence of the authors Dr. Taylor's paper was presented by Professor Osgood, and the papers of Professors Snyder, Carver, Miller, Fite, and Safford were read by title. Abstracts of the papers follow below.

1. In his vice-presidential address before the American Asso-

ciation for the Advancement of Science (1886) Gibbs suggested a method for obtaining the results of Grassmann's linear algebra by the use of what he called an indeterminate product. Following this method Dr. Phillips gives proofs of a number of the fundamental theorems of Grassmann.

2. If the lines of a polygon of $2n$ sides lie on a quadric surface, it follows from a theorem of Poncelet that the lines joining the odd vertices to the non-adjacent even ones lie on a surface of class $n-2$. Dr. Phillips discusses these surfaces and the configurations of lines for the cases $n=5$ and $n=6$.

3. In Professor Snyder's paper a method was developed for determining all the congruences formed by the generators of a bundle of quadric surfaces, and for distinguishing the necessary and sufficient condition that the two systems generate congruences that are rationally separable. The lines l, l' through a basic point B_i of a bundle of quadrics $\Sigma\lambda_i H_i$ are arranged in pairs, each of which uniquely determines the other, defining an involution I. A cone K with vertex at B_i fixes a congruence σ in Σ and its image K' in I fixes another congruence τ; the lines of σ, τ are formed by conjugate systems of generators of the same family of quadrics. The singular cones at B_j, B_k are birationally equivalent. The congruences σ, τ define an infinite discontinuous birational group of point transformations which leave their common focal surface invariant. Finally, both congruences define a relation $f(\lambda)=0$ which is the equation of a contact curve of the discriminant $\Delta(\lambda)$ in the λ plane. All the contact curves of $\Delta(\lambda)$ having an odd characteristic can be obtained in this way.

4. In his Vorlesungen über Zahlentheorie, Sommer shows that any ideal of the quadratic number field $k(\sqrt{m})$ can be reduced to the canonic form $(i, i_1 + i_2\omega)$, where i, i_1, and i_2 are rational integers, i_2 a factor of i and i_1, and ω is $\sqrt{m}$ or $\frac{1}{2}(1+\sqrt{m})$ according as $m \not\equiv 1 \pmod 4$ or $m \equiv 1 \pmod 4$. In the present paper Dr. Carver uses a slightly modified canonic form, viz., $r(s, t+\omega)$, r, s and t being rational integers and ω having the meaning given above, with the conditions $s > t \geqq 0$ and

$$t^2 \equiv m \pmod s, \qquad \text{if } m \not\equiv 1 \pmod 4;$$

$$(2t+1)^2 \equiv m \pmod{4s}, \qquad \text{if } m \equiv 1 \pmod 4.$$

A method is developed for factoring, multiplying, and dividing (when division is possible) ideals in this canonic form.

5. If H represents any subgroup of a group G, all the operators of G may be represented, without repetition, in either of the following two ways:

$$G = H + HS_2 + HS_3 + \cdots + HS_\gamma$$
$$= H + T_2H + T_3H + \cdots + T_\gamma H.$$

If for each HS_α ($\alpha = 2, 3, \cdots, \gamma$) it is possible to find some $T_\beta H$ so that all the operators of HS_α coincide with those of $T_\beta H$, then H is an invariant subgroup of G, and vice versa. Galois called attention to this important case and named it a proper decomposition of G. Professor Miller considers the general case where HS_α has exactly ρ operators in common with $T_\beta H$ and proves that, if this condition is satisfied, the operators of both HS_α and $T_\beta H$ transform H into a group which has exactly ρ operators in common with H. In particular, if all the operators of HS_α coincide with those of $T_\beta H$ then H is invariant under each of the operators of HS_α. He also proves that in any group whatsoever it is possible to select the operators $S_2, S_3, \cdots, S_\gamma$ in such a way that all the operators of G may be represented, without repetition, in either of the following ways, H being any arbitrary subgroup of G:

$$G = H + HS_2 + HS_3 + \cdots + HS_\gamma$$
$$= H + S_2H + S_3H + \cdots + S_\gamma H.$$

In Weber's Lehrbuch der Algebra, volume 2, 1896, page 8, it is observed that the second of these decompositions is also equal to

$$H + S_2^{-1}H + S_3^{-1}H + \cdots + S_\gamma^{-1}H.$$

Hence the S's may also be replaced by their inverses without affecting either decomposition as a whole.

6. The solution of the problem of mapping the interior of a simply connected region S on the interior of a circle was completed by Professor Osgood in his proof of the existence of the Green's function of the most general simply connected plane region. The question as to whether the boundary of S, when this boundary is of the most general nature, will be transformed

continuously into the circumference of the circle, was answered by Professor Osgood in a set of theorems published in volume 9 of the BULLETIN. The major part of Dr. Taylor's paper is given to the proofs of these theorems.

7. In view of the erroneous statement by Reye in his Geometrie der Lage that a space collineation A with just four invariant points can be commutative only with collineations that have the same invariant points, Professor Fite determines in detail what must be the invariant points of collineations commutative with A. He extends the results for three dimensional space to collineations in space of $n-1$ dimensions, and determines the nature and number of the collineations that are commutative with both A and B, B being any collineation commutative with A.

8. Professor Richardson shows that the minimum of the function $f(x, y, z)$ for those values of the variables that satisfy the relation $g(x, y, z) + \pi h(x, y, z) = 0$ is a function $M(\pi)$ of the parameter π, which when maximized gives the same constant as the minimum of the function $f(x, y, z)$ for those values of the variables which satisfy the relations $g(x, y, z) = 0$, $h(x, y, z) = 0$.

In the calculus of variations the two problems corresponding are also found to be equivalent:

1. To find a function $y(x)$ which satisfies the boundary conditions

$$(A) \qquad y(0) = y(1) = 0$$

and the integral relations

$$\int_0^1 g(x, y, dy/dx)dx = 0, \quad \int_0^1 h(x, y, dy/dx) = 0,$$

and minimizes the integral

$$(B) \qquad \int_0^1 f(x, y, dy/dx).$$

2. The minimum of the integral (B) for those functions $y(x)$ which satisfy the boundary conditions (A) and the integral condition

$$\int_0^1 \{g(x, y, dy/dx) + \pi h(x, y, dy/dx)\}dx = 0$$

is a function of the parameter π. To determine π in such a way that the minimum is maximized.

In both theories the results admit of immediate generalization.

9. Moore and Wiman have determined the subgroups of the linear fractional group $LF(2, p^n)$. Mr. Mitchell gives another method for the determination of those subgroups which contain additive groups. It is based on the solution of the Diophantine equation

$$\Omega = 1 + (p^m - 1)\frac{\Omega}{d_1 p^{m_1}} + \sum_{i=1}^{r}(d_i - 1)\frac{\Omega}{f_i d_i} \quad (f_i = 1, 2).$$

10. In this paper Mr. Mitchell extends the results obtained by him concerning the subgroups of the linear group $LF(3, p)$, to the more general group $LF(3, p^n)$. The group is represented as a collineation group in the finite plane $PG(2, p^n)$.

The maximal subgroups are: the $LF(3, p^k)$, where k is a divisor of n; the hyperorthogonal groups $HO(3, p^k)$, which appear for n even and k a factor of $\frac{1}{2}n$; groups leaving invariant a point, line, triangle, and conic; a G_{720} (for $p = 5$), $G_{\frac{1}{2}7!}$ ($p = 5$), the G_{360}, G_{216}, and G_{168}.

11. Professor Moore makes use of the ten coordinates of a line in S_4 to discuss some properties of five-parameter families which involve second and higher differentials. The interpretation in the geometry of circles in ordinary space is then given.

12. The forces considered by Professor Kasner lead to differential equations of the form $\ddot{x} = \phi(x, y, t)$, $\ddot{y} = \psi(x, y, t)$. In connection with the trajectories (xy curves), it is of interest to introduce the xt curves and the yt curves, t being represented as a space coordinate. The transformations of x, y, t converting a pair of equations of the above type into one of the same type form an infinite group involving three arbitrary functions.

13. In Professor Safford's paper Sturm's method of integrating $dx/\sqrt{X} + dy/\sqrt{Y} = 0$ is discussed for a somewhat more general form of X than usual. The integrating factor is found as a function of xy and the coefficients of X. The determination of this factor has previously been in most cases dependent upon a knowledge of the integral to be obtained instead of upon the given differential equation.

14. By interpreting the projective geometry within a linear line complex as a geometry of line elements in the plane by

means of a transformation of Lie, Dr. Gundelfinger effects a classification of ordinary differential equations of the first order with respect to the arrangement of the ∞^2 osculating circles to their integral curves and develops a theory of reciprocal line element loci.

F. N. COLE,
Secretary.

THE APRIL MEETING OF THE CHICAGO SECTION.

THE twenty-seventh regular meeting of the Chicago Section of the AMERICAN MATHEMATICAL SOCIETY was held at the University of Chicago on Friday and Saturday, April 8–9, 1910. Professor L. E. Dickson, Vice-President of the Society and Chairman of the Section, presided at the three sessions held on Friday morning and afternoon and Saturday morning. The attendance at the various sessions included twenty visitors and the following forty-four members of the Society:

Mr. R. P. Baker, Mr. W. H. Bates, Professor G. A. Bliss, Professor Oskar Bolza, Dr. R. L. Börger, Dr. H. E. Buchanan, Dr. Thomas Buck, Dr. H. T. Burgess, Dr. A. R. Crathorne, Professor D. R. Curtiss, Professor L. E. Dickson, Dr. Arnold Dresden, Professor W. B. Ford, Professor A. G. Hall, Professor E. R. Hedrick, Mr. T. H. Hildebrandt, Professor O. D. Kellogg, Professor Kurt Laves, Dr. A. C. Lunn, Dr. W. D. MacMillan, Mr. E. J. Miles, Professor G. A. Miller, Dr. R. L. Moore, Professor E. H. Moore, Professor J. C. Morehead, Professor F. R. Moulton, Dr. L. I. Neikirk, Dr. Anna J. Pell, Professor Alexander Pell, Professor H. L. Rietz, Miss Ida M. Schottenfels, Mr. A. R. Schweitzer, Professor J. B. Shaw, Mr. R. R. Shumway, Professor C. H. Sisam, Professor H. E. Slaught, Professor E. J. Townsend, Professor A. L. Underhill, Professor E. B. Van Vleck, Professor E. J. Wilczynski, Professor B. F. Yanney, Professor J. W. Young, Professor J. W. A. Young, Professor Alexander Ziwet.

On Friday evening nearly all of the members present at the meeting dined together at the Quadrangle Club, at which time the question of holding the next meeting of the Section at Minneapolis was discussed. A letter was read from the President of the University of Minnesota urging the Section to meet

there in connection with the annual convocation of the American Association for the Advancement of Science. Both the president and secretary of Section A of the Association, Professors E. H. Moore and G. A. Miller, urged the desirability of closer affiliation between the mathematicians and other scientists, especially the astronomers and physicists. It was finally voted to hold the December meeting of the Section at Minneapolis, unless the Council should decide that the annual meeting of the Society, with the presidential address, will be held at Chicago at that time. It had been previously announced that the Council had considered favorably the question of holding a meeting of the entire Society at Chicago either in December, 1910, or in April, 1911, depending upon the convenience of the President, Professor Bôcher.

The following papers were read at this meeting:

(1) Professor D. R. CURTISS: "An extension of Descartes's rule of signs."

(2) Professor C. H. SISAM: "On a class of r-spreads in space of n dimensions."

(3) Dr. ANNA J. PELL: "On a functional equation."

(4) Dr. H. E. BUCHANAN: "The equations of variation for the straight line elliptical orbits of three finite masses."

(5) Professor J. B. SHAW: "On quaternions."

(6) Professor W. B. FORD: "On the determination of the asymptotic developments of a given function (second paper)."

(7) Professor OSKAR BOLZA: "An application of 'general analysis' to a problem in the calculus of variations."

(8) Dr. H. T. BURGESS: "Differential equations of the projective curves on a quadric whose tangents belong to a line complex."

(9) Professor E. J. WILCZYNSKI: "On the general theory of congruences of straight lines."

(10) Mr. R. P. BAKER: "On a class of equations of state representing normal and abnormal three-state bodies."

(11) Dr. L. I. NEIKIRK: "A theorem on (m, n) correspondence."

(12) Mr. W. H. BATES: "Note on the generalization of the formulas of Gauss and Codazzi."

(13) Mr. W. H. BATES: "On the medium curvature of a hypersurface (second paper)."

(14) Professor G. A. MILLER: "Groups of transformations of Sylow subgroups."

(15) Professor G. A. MILLER: "Extensions of theorems due to Cauchy."

(16) Dr. R. L. MOORE: "On Duhamel's theorems."

(17) Miss IDA M. SCHOTTENFELS: "The Fano geometry."

(18) Professor F. R. MOULTON: "On the problem of the spherical pendulum."

(19) Professor L. E. DICKSON: "Equivalence of families of quadratic forms."

Professor Dickson's paper was read by title. Professor Bolza's paper appeared in full in the May number of the BULLETIN. Abstracts of the other papers follow below, the numbers corresponding to those in the above list.

1. Descartes's rule of signs gives merely an upper limit for the number of positive roots of an algebraic equation $f(x) = 0$. To determine the exact number of such roots we must usually have recourse to other methods. In Professor Curtiss's paper it is, however, shown that there always exists a polynomial $\phi(x)$ such that the number of variations of sign in the coefficients of $F(x) = f(x)\phi(x)$ is exactly equal to the number of positive roots of $f(x) = 0$. Further, $\phi(x)$ can be so chosen that $F(x)$ has only $m + 1$ coefficients, where m is the degree of $f(x)$, and these coefficients are determinants of a matrix formed from the coefficients of $f(x)$. From this is derived a criterion for the number of real roots in any interval (a, b). The paper concludes with a graphical method for applying these tests.

2. The principal theorem in Professor Sisam's paper is the following: If V_r is an analytic r-spread belonging to a space of n dimensions ($n \geqq r + 1$); if through a generic point of V_r there does not pass a continuum of spaces of t dimensions lying on V_r; and if the section of V_r by a generic hyperplane through the tangent r-dimensional space at a generic point P of V_r is an $(r - 1)$-spread the tangent cone to which at P has a double t-dimensional space; then V_r is generated by t-dimensional spaces in such a way that the tangent r-dimensional space to V_r is invariant along each t-dimensional space.

3. The functional equation $\lambda\phi(s) = T[\phi(s)]$, where T is a single-valued fractional transformation with the two properties

$$(f) \qquad \int_a^b f(s)T[f(s)]\,ds \geqq 0,$$

$$(f_1, f_2) \qquad \int_a^b f_1(s)T[f_2(s)]\,ds = \int_a^b f_2(s)T[f_1(s)]\,ds,$$

is reduced by means of a biorthogonal system of functions to an infinite system of linear equations whose coefficients form a limited symmetric matrix. Dr. Pell considers the character of solutions corresponding to a point, or continuous spectrum of λ, and finds conditions for the solvability of the non-homogeneous functional equation, together with some expansion theorems.

4. In his dissertation Dr. Buchanan discussed some periodic orbits of three bodies near the straight line circular solutions. The present paper is a report on the extension to the straight line elliptical solutions. For $e = 0$, the solutions obtained here reduce to those obtained in the circular case. The equations of variation, formed in the usual way, are a system of homogeneous linear equations of the twelfth order, but two of them are independent of the other four. The coefficients of the equations are expanded as power series in the eccentricity, the coefficients of these power series being simply periodic. The solutions are found to exist as power series in e. The characteristic exponents are found to be uniquely determined by the periodicity conditions and to have the same character as in the circular case.

5. In this paper Professor Shaw develops formulas for certain functions of two, three, four, or more quaternions, by means of which the various differentiating operators of quaternions may be readily expressed. Thus formulas are found for the quaternion operator

$$\frac{\partial}{\partial w} + i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}$$

which are as manageable as those for ∇. Theorems of integration are given. These functions furnish an application to quaternions of general formulas previously given in the author's papers on general associative algebra.

6. In Professor Ford's paper a method is derived for determining the asymptotic developments of a given function, and application is made to a number of important functional types. The results supplement those of a former paper (see April number of the *Annals of Mathematics*); also those of Barnes, Lindelöf, Wiman, Mattson, and others concerning the asymptotic develments of integral functions defined by canonical (Weierstrass)

products. A new type of development is obtained for these latter functions, thus enabling one to study their behavior for large values of the variable in regions which were precluded in the previous developments.

8. Dr. Burgess shows that the differential equation of the curves, upon the quadric $\phi(x) \equiv x_1x_4 + x_2x_3 = 0$, whose tangents belong to the line complex $F(p_{12}, p_{13}, p_{14}, p_{34}, p_{42}, p_{23}) = 0$, is homogeneous in six arguments and of the form

$$F(u^2dv, v^2du, -udv - vdu, -dv, du, vdu - udv) = 0,$$

where the parametric equations of the quadric ϕ are

$$\rho x_1 = uv, \quad \rho x_2 = u, \quad \rho x_3 = v, \quad \rho x_4 = -1.$$

The case for a *linear* line complex is then studied in detail and the theory of elementary divisors is applied to get the canonical forms of the differential equations, together with their integrals. Next, the case where $F(p) = 0$ is quadratic is attacked in the particular form in which $F(p) = 0$ is the line equation of a quadric surface. Here the theory of elementary divisors is again applied and the canonical forms of the differential equations are found.

9. The new theory of congruences presented by Professor Wilczynski is based upon the consideration of the completely integrable system of partial differential equations

$$(D) \qquad \begin{aligned} y_v &= mz, \quad z_u = ny, \\ y_{uu} &= ay + bz + cy_u + dz_v, \\ z_{vv} &= a'y + b'z + c'y_u + d'z_v. \end{aligned}$$

The integral surfaces of (D) are the two sheets of the focal surface of the congruence, the curves $u =$ const. and $v =$ const. upon these surfaces being their intersections with the developables of the congruence. The lines of the congruence are obtained by joining corresponding points (u, v) of the two surfaces. All of the known results in the general theory of congruences, so far as they are of a projective nature, present themselves in a very simple way from this point of view, and a large number of new theorems make their appearance. The so-called Laplace transformation which was first associated with the theory of congruences by Darboux appears in a new light, enabling one

to formulate and solve far more general problems than has hitherto been possible. The general theory is applied to two particular problems: first, to the determination of those congruences which belong to a linear complex and for which the first Laplace transformation gives rise to a congruence that likewise belongs to a linear complex; second, to the study of those congruences whose focal surfaces are quadrics. Both of these problems are closely connected with the differential equation

$$\frac{\partial^2\theta}{\partial u\partial v} = e^{2\theta} - e^{-2\theta}.$$

10. The equations considered in Mr. Baker's paper are of the form $p = tF - TM$, in which F represents the gas law and is explicitly discussed for Van der Waal's form $1/(v-1)$ and for Planck's form $-\log(1-1/v)$; $M = 1/v^2 - c/v^3 + d/v^4$ and is an extension of Van der Waal's term which may ultimately be a power series; and $T = mt/(t^2 + kt + l^2)$ is a special form of Amagat's multiplier. The conditions for the existence of a triple point when $T = 1$ are worked out for both the gas laws mentioned and give in each case an area in the (c, d) plane within which triple points are possible. If $k > 0$, the surface is of the same type as for a normal three state body, the LS line ending at a critical point or not, according to the dependence of c, d on the temperature, quite small movements of the (c, d) point deciding. If $k < 0$ and the (c, d) point passes out of the triple point area before the critical temperature is reached, the equation represents the type of an abnormal body. If this does not happen, a form approximating the "Tamann idea" is reached, though the complete form requires seven values of v for given p, t. The particular form of the gas law is indifferent as far as type is concerned; and the reversal of sign of the molecular pressure is neither necessary nor sufficient for abnormality, but may occur with a very low temperature for the triple point. By a proper choice of the three scales and the constants, the equations may be compared with the observations.

11. Emil Weyr in a paper * published in 1870 defines an n-fold involution of points on a straight line as the points whose abscissas are roots of the equation $f(x) - \lambda\phi(x) = 0$, where the base functions $f(x)$ and $\phi(x)$ are rational integral functions of

* *Math. Annalen*, vol. 3 (1870), pp. 34–44.

degree n, and λ is an arbitrary parameter. He finds its double elements and other properties. In a second paper† he introduces the two involutions $f(x) - \lambda\phi(x) = 0$ and $F(y) - \mu\psi(y) = 0$, of degrees m and n, and calls these involutions projective when $A\lambda\mu + B\lambda + C\mu + D = 0$. He considers the plane curves defined by the projective involutions and develops their properties. Dr. Neikirk considers an (m, n) correspondence given by a rational integral equation $f(x, y) = 0$ of degree m in x and n in y which has at least a single set of correspondences of the type $(x_1, x_2, x_3 \cdots, x_m ; y_1, y_2, y_3 \cdots, y_n)$ and shows that $f(x, y) = 0$ may be reduced to $\theta(x) + \lambda\phi(x) = 0, \chi(y) + \lambda\psi(y) = 0$, where θ, ϕ, χ, and ψ are rational integral functions, the first two of degree m, the last two of degree n, and λ is a variable parameter.

12. In his first paper Mr. Bates uses Maschke's symbolic notation in generalizing the ordinary method of deriving the Gauss and Codazzi formulas.

13. In a paper read before the Society in December, 1909, Mr. Bates expressed those medium curvatures of R_n that have even subscripts, in terms of the first fundamental quantities of R_n. In the present paper, he makes use of a theorem of Maschke to express those medium curvatures of R_{n-1} that have odd subscripts, in terms of the first fundamental quantities of R_n and the function which defines R_{n-1} in R_n. The formula for the first medium curvature K_1 contains as special cases the curvature of ordinary plane and twisted curves and the medium curvature of the surface $f(x, y, z) = 0$.

14. Professor Miller proves the following theorems: The necessary and sufficient condition that a transitive group of degree n is a group of transformations of Sylow subgroups is that its subgroup composed of all its substitutions which omit a given letter involves one and only one Sylow subgroup of degree $n - 1$. Hence there are only two alternating and also only two symmetric groups which are groups of transformations of Sylow subgroups. If the Sylow subgroups of order p^m in a group are transformed under this group according to an imprimitive group, the number of these Sylow subgroups is of the form $(1 + k_1 p^\alpha)(1 + k_2 p^\alpha)$ where none of the integers k_1, k_2, α is zero. Moreover, the systems of imprimitivity must be transformed

† *Prager Sitzungsberichte*, 1870, pp. 14–19.

according to some non-regular group. If a group involves only 3 Sylow subgroups of order 2^m these subgroups generate a group of order $3 \cdot 2^m$; if it involves exactly 4 Sylow subgroups of order 3^m they must generate a group whose order is either $4 \cdot 3^m$ or $8 \cdot 3^m$; if it involves exactly 6 Sylow subgroups of order 5^m they generate a group whose order is either $12 \cdot 5^m$ or $24 \cdot 5^m$. The group generated by these Sylow subgroups is a characteristic subgroup of the entire group if it does not coincide with it.

15. The object of Professor Miller's second paper was to extend a theorem due to Cauchy and to indicate how Sylow's theorem is an almost immediate result of this extended theorem, which was stated as follows: The number of those operators of a group G, involving H_1, H_2 as subgroups, which transforms H_1 into a group having exactly ρ operators in common with H_2 is divisible by $h_1 h_2/\rho$, h_1 and h_2 being the orders of H_1 and H_2 respectively. This theorem was proved by Cauchy for the special case where $\rho = 1$, and this special case was used by him to prove that every group whose order is divisible by a given prime number p must involve operators of order p. The extended theorem may be used, in a similar way, to prove that every group whose order is divisible by p^α must involve a subgroup of order p^α.

16. Dr. Moore discusses certain conditions under which the limit of the sum of one set of infinitesimals is the same as that of another set.

17. In the *Annals of Mathematics*, second series, volume 11, number 2, page 60, Mr. George M. Conwell discusses the Fano configuration and its group. Miss Schottenfels in the present paper discusses the postulates under which the Fano geometry was developed and its extension to n-space. See *Giornali di Matematiche*, volume 30 (1892).

18. The problem of the spherical pendulum gives rise to the differential equations

$$\frac{d^2x}{dt^2} + (a_0 + a_1 z)x = 0, \quad \frac{d^2y}{dt^2} + (a_0 + a_1 z)y = 0,$$

$$\frac{d^2z}{dt^2} + (b_0 + b_2 z^2)z - g = 0,$$

where x, y and z are subject to the conditions

$$x^2 + y^2 + z^2 = l^2.$$

Professor Moulton treats the third equation as one of a class possessing periodic solutions. The existence of the solutions is proved and practical methods of constructing them as power series in a parameter are given. Then the first two equations become homogeneous linear differential equations having simply periodic coefficients of the form first treated by Hill in his memoir on the motion of the lunar perigee. A distinct treatment of these equations is given, both as to the existence of the solutions and as to practical methods of finding them.

19. Professor Dickson discusses the equivalence of two pencils of quadratic forms in n variables. If in the pencil $A = \lambda_1 A_1 + \lambda_2 A_2$, we replace the generators A_1 and A_2 by $A_1 = gA_1 + hA_2$ and $\bar{A}_2 = g'A_1 + h'A_2$, where $gh' - g'h \neq 0$, then $\bar{A} = \lambda_1 \bar{A}_1 + \lambda_2 \bar{A}_2$ becomes $\bar{\lambda}_1 A_1 + \bar{\lambda}_2 A_2$, where $\bar{\lambda}_1 = \lambda_1 g + \lambda_2 g'$, $\bar{\lambda}_2 = \lambda_1 h + \lambda_2 h'$. An elementary divisor $(a_1\lambda_1 + a_2\lambda_2)^s$ of $|\bar{A}|$ corresponds to the elementary divisor $(\bar{a}_1\lambda_1 + \bar{a}_2\lambda_2)^s$ of $|\bar{A}|$, where $\bar{a}_1 = a_1 g + a_2 h$, $\bar{a}_2 = a_1 g' + a_2 h'$. It follows that two non-singular pencils of quadratic forms are equivalent if and only if the elementary divisors of one pencil can be derived from those of the second by a linear transformation on the two parameters.

To discuss the problem of equivalence in a given field F, use is made of the results in *Transactions*, volume 10 (1909), pages 347–360. For the above replacement of generators, the elementary divisor $(\lambda - c)^s$ of $|\lambda A_1 - A_2|$ corresponds to the elementary divisor $(\lambda - \gamma)^s$ of $|\lambda \bar{A}_1 - \bar{A}_2|$, where $\gamma = (g' + ch')/(g + ch)$. Hence from the ultimate canonical types (obtained in the paper cited) of non-singular pairs of quadratic forms we may derive canonical types of pencils of forms by restricting the roots $c_1, c_2, \cdots, c_m$ to sets not equivalent under linear fractional transformation in the initial field F.

H. E. SLAUGHT,
Secretary of the Section.

GROUPS GENERATED BY TWO OPERATORS EACH OF WHICH IS TRANSFORMED INTO A POWER OF ITSELF BY THE SQUARE OF THE OTHER.

BY PROFESSOR G. A. MILLER.

(Read before the Chicago Section of the American Mathematical Society, January 1, 1910.)

§ 1. *Introduction.*

Two special cases of the category of groups defined by the heading of this paper have been considered; viz., when the square of each of the two generators transforms the other generator either into itself* or into its inverse.† It was observed that in the former of these two cases the orders of the two generators are not restricted, while in the latter each of these orders must divide 8. Each of these special cases led to a very elementary category of solvable groups. It will be proved that the more general category defined by the heading of this paper is also composed entirely of solvable groups of simple structure.

As an instance of how such generalizations may lead to very complex categories of groups we may give the theorem that *every symmetric group can be generated by two operators whose squares are commutative.* In fact, the symmetric group of degree n is evidently generated by the following two cyclic substitutions whose squares are commutative:

$$t_1 = (x_1 x_2 x_3 \cdots x_{n-1}), \quad t_2 = (x_1 x_n).$$

From the theorem that every symmetric group whose degree exceeds 8 can be generated by two substitutions of orders 2 and 3 respectively ‡ it results directly that all such groups are included in the category of groups defined by the condition that each of them can be generated by two operators which are transformed into themselves by the square and cube respectively of the other.

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* BULLETIN, vol. 16 (1910), p. 173.
† *Annals of Mathematics*, vol. 9 (1907), p. 48.
‡ BULLETIN, vol. 7 (1901), p. 426.

§ 2. *General Considerations.*

The conditions imposed upon the two generators s_1, s_2 of the group G, as expressed by the heading of this paper, give rise to the following equations:

$$s_1^{-2}s_2s_1^2 = s_2^{\alpha}, \quad s_2^{-2}s_1s_2^2 = s_1^{\beta}.$$

If at least one of the two numbers α, β is even, the corresponding operator is of odd order and hence it must be generated by its square. In this case G is generated by a cyclic group and an operator transforming this cyclic group into itself. As many properties of these groups are well known, we shall confine our attention, in what follows, to the consideration of cases in which both α and β are odd. The group H generated by s_1^2, s_2^2 clearly belongs to the elementary category of groups which may be generated by two operators each of which transforms the other into a power of itself.* Hence the commutator subgroup of H is cyclic and the order of H divides the quotient obtained by dividing the product of the orders of s_1^2, s_2^2 by the order of the commutator subgroup.

It is easy to see that H is invariant under G, since α and β are odd and the following equations are satisfied:

$$s_1^{-1}s_2^2s_1 = s_2^2s_2^{-2}s_1^{-1}s_2^2s_1 = s_2^2s_1^{1-\beta}, \quad s_2^{-1}s_1^2s_2 = s_1^2s_2^{1-\alpha}.$$

From the fact that

$$s_2^{-2}s_1^{-2}s_2^2 = s_2^{2\alpha-2}s_1^{-2} = s_1^{-2\beta}$$

it results that

$$s_1^{2(\beta-1)}s_2^{2(\alpha-1)} = 1.$$

Hence the two operators $s_1^{2(\beta-1)}$, $s_2^{2(\alpha-1)}$ are invariant under G. To find multiples of the orders of s_1, s_2 we may transform these invariant operators by $s_2^2s_1^2$ respectively, as follows:

$$s_2^{-2}s_1^{2(\beta-1)}s_2^2 = s_1^{2\beta(\beta-1)} = s_1^{2(\beta-1)}, \quad \text{or} \quad s_1^{2(\beta-1)^2} = 1;$$

$$s_1^{-2}s_2^{2(\alpha-1)}s_1^2 = s_2^{2\alpha(\alpha-1)} = s_2^{2(\alpha-1)}, \quad \text{or} \quad s_2^{2(\alpha-1)^2} = 1.$$

Hence the theorem: *If two operators s_1, s_2 satisfy the equations $s_1^{-2}s_2s_1^2 = s_2^{\alpha}$, $s_2^{-2}s_1s_2^2 = s_1^{\beta}$, their orders divide $2(\beta-1)^2$ and $2(\alpha-1)^2$ respectively, their squares generate a group which involves a cyclic commutator subgroup and is invariant under the*

* *Quar. Jour. of Math.*, vol. 37 (1906), p. 286.

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It is easy to see that H is invariant under G, since α and β are odd and the following equations are satisfied:

$$s_1^{-1}s_2^2s_1 = s_2^2s_2^{-2}s_1^{-1}s_2^2s_1 = s_2^2s_1^{1-\beta}, \quad s_2^{-1}s_1^2s_2 = s_1^2s_2^{1-\alpha}.$$

From the fact that

$$s_2^{-2}s_1^{-2}s_2^2 = s_2^{2\alpha-2}s_1^{-2} = s_1^{-2\beta}$$

it results that

$$s_1^{2(\beta-1)}s_2^{2(\alpha-1)} = 1.$$

Hence the two operators $s_1^{2(\beta-1)}$, $s_2^{2(\alpha-1)}$ are invariant under G. To find multiples of the orders of s_1, s_2 we may transform these invariant operators by $s_2^2s_1^2$ respectively, as follows:

$$s_2^{-2}s_1^{2(\beta-1)}s_2^2 = s_1^{2\beta(\beta-1)} = s_1^{2(\beta-1)}, \quad \text{or} \quad s_1^{2(\beta-1)^2} = 1;$$

$$s_1^{-2}s_2^{2(\alpha-1)}s_1^2 = s_2^{2\alpha(\alpha-1)} = s_2^{2(\alpha-1)}, \quad \text{or} \quad s_2^{2(\alpha-1)^2} = 1.$$

Hence the theorem: *If two operators s_1, s_2 satisfy the equations $s_1^{-2}s_2s_1^2 = s_2^{\alpha}$, $s_2^{-2}s_1s_2^2 = s_1^{\beta}$, their orders divide $2(\beta-1)^2$ and $2(\alpha-1)^2$ respectively, their squares generate a group which involves a cyclic commutator subgroup and is invariant under the*

* *Quar. Jour. of Math.*, vol. 37 (1906), p. 286.

group generated by s_1 and s_2, and they satisfy the equation $s_1^{2(\beta-1)} = s_2^{2(1-\alpha)}$.

The quotient group of G with respect to H is dihedral, since it can be generated by two operators of order 2. As H is solvable and this quotient group is solvable, G is always solvable. That is, *if two operators are such that each is transformed into a power of itself by the square of the other, they generate a solvable group whose fourth derived is identity.* As an instance of such a group, we may cite the symmetric group of order 24. This is generated by any two of its operators of order 4 which do not have a common square, and each of these two operators is transformed into its inverse by the square of the other. In this case the third derived is already identity. To obtain a group whose third derived is not identity we may consider the group of order 48 generated by two operators of order 8 each of which is transformed into its inverse by the square of the other.* This group of order 48 illustrates also that the orders of s_1, s_2 may be actually $2(\beta-1)^2$, $2(\alpha-1)^2$ respectively.

The theorem stated above implies that the numbers α, β always fix an upper limit for the orders of s_1, s_2 except when at least one of these numbers is unity. When both of them are unity there results the special case noted in the introduction. If only one of them (α) is unity the defining relations assume the form

$$s_1^{-2}s_2s_1^2 = s_2, \quad s_2^{-2}s_1s_2^2 = s_1^\beta \quad (\beta \neq 1).$$

Hence $2 = 2\beta + kn$, n being the order of s_1 and $k \neq 0$. That is, if we assume $\alpha = 1$, $\beta \neq 1$, it results that the order of s_1 is a divisor of $2(\beta-1)$. These conditions do not fix an upper limit for the order of s_2, as may be seen from the following special case. The two operators s_1, s_2 may evidently be the generators of the dicyclic group of order 16, s_2 generating the cyclic group of order 8 and β being 3. If s_2 is multiplied by an operator of arbitrary order which is commutative with each of the operators s_1, s_2, the product thus obtained and s_1 will again satisfy the given conditions. This proves that the order of s_2 may be an arbitrary multiple of 8 when $\alpha = 1$ and $\beta = 3$, but the order of s_1 must be 4.

It was observed above that the orders of s_1, s_2 divide $2(\beta-1)^2$, $2(\alpha-1)^2$ respectively and that the $2(\beta-1)$th power of the former of these operators is equal to the $2(1-\alpha)$th power of the

* *Annals of Mathematics*, vol. 9 (1907), p. 51.

latter. Hence it results that the order of each of these operators is a divisor of $2(\alpha-1)(\beta-1)$. That is, *if two operators are such that each is transformed into a power of itself by the square of the other, the order of each of them is a divisor of twice the product of the indices of these powers diminished by unity.* When α and β are different this evidently furnishes a lower limit for the order of one of the operators than the one given before. Since $s_1^{2(\beta-1)}$ and $s_2^{2(\alpha-1)}$ are invariant under G, it results that

$$s_1^{\beta^{\alpha-1}} = s_1, \quad s_2^{\alpha^{\beta-1}} = s_2.$$

Hence the order of s_1 is a divisor of $\beta^{\alpha-1}-1$, and that of s_2 is a divisor $\alpha^{\beta-1}-1$. These conditions will sometimes give lower limits for these orders than those given above, as may be seen from the special cases in the following section.

The preceding considerations prove that two operators which satisfy the two conditions

$$s_1^{-2}s_2s_1^2 = s_2^\alpha, \quad s_2^{-2}s_1s_2^2 = s_1^\beta$$

have their orders limited by these conditions when, and only when, each of the numbers α, β is different from unity. When α, β satisfy this condition, s_1^2, s_2^2 generate an invariant subgroup whose order is a divisor of the product of their orders. For every pair of odd values for α, β there is an infinite system of groups whose fourth derived must be unity and each of these systems must include the dihedral groups. The properties of the invariant subgroup generated by s_1^2, s_2^2 are known, since these operators transform each other into powers, and every G may be constructed by adjoining to this invariant subgroup the operator s_1s_2 and then adjoining to the invariant subgroup thus obtained the operator s_1. While these fundamental properties apply to every possible G, a number of interesting special properties apply to given values of α and β, as may be seen from the examples of the following section.

§ 3. *Special Cases.*

The following special cases may serve to illustrate some of the preceding theorems. Suppose that s_1, s_2 are any two operators which satisfy the two equations

$$s_1^{-2}s_2s_1^2 = s_2^3, \quad s_2^{-2}s_1s_2^2 = s_1^3.$$

From the general results of the preceding section it follows that $s_1^4 = s_2^4$, $s_1^8 = s_2^8 = 1$, and that s_1, s_2 have the same order when-

ever neither of these operators is identity. When s_1, s_2 are commutative they generate the four-group or a subgroup of this group. This trivial case will be excluded in what follows. That is, we shall assume that s_1, s_2 are non-commutative. When each of these operators is of order 2 they generate a dihedral group and every dihedral group may evidently be generated by two operators satisfying the given conditions. The only two possible cases which remain to be considered are when the common order of s_1, s_2 is either 4 or 8. In each of these cases the order of $s_1 s_2$ is a multiple of 3 since

$$s_1^{-1}s_2^{-1}\cdot s_2^2\cdot s_2 s_1 = s_1^{-1}s_2^2 s_1 = s_2^2\cdot s_2^{-2}s_1^{-1}s_2^2\cdot s_1 = s_2^2 s_1^{-2},$$

$$s_1^{-1}s_2^{-1}\cdot s_2^2 s_1^{-2}\cdot s_2 s_1 = s_1^{-1}s_2 s_1^{-2}s_2 s_1 = s_1^{-3}\cdot s_1^2 s_2 s_1^{-2}\cdot s_2 s_1 = s_1^{-2}s_2^4 = s_1^2,$$

$$s_1^{-1}s_2^{-1}\cdot s_1^2\cdot s_2 s_1 = s_1\cdot s_1^{-2}s_2^{-1}s_1^2\cdot s_2 s_1 = s_1 s_2^{-2}s_1 = s_2^{-2}\cdot s_2^2 s_1 s_2^{-2}s_1 = s_2^2.$$

When the common order of s_1, s_2 is 4 the invariant subgroup H generated by s_1^2, s_2^2 is the four-group. If the order of $s_1 s_2$ is 3, G must be the symmetric group of order 24. In fact, in this case G is generated by two operators which are such that each is transformed into its inverse by the square of the other and hence its properties are known, as was observed in the introduction. The only case which remains to be considered is the one where the common order of s_1, s_2 is 8. In this case H is the quaternion group and the order of G is a multiple of 48. When the order of s_1, s_2 is 3, G is one of the four groups of order 48 which involve the non-twelve group of order 24. Since $(s_1 s_2^2)^2 = s_1 s_2^2 s_1 s_2^2 = s_1 s_2^4\cdot s_2^{-2}s_1 s_2^2 = 1$, this group can be represented as a transitive substitution group on 8 letters. It is easy to verify that the following generators of this transitive substitution group satisfy the conditions imposed upon s_1, s_2:

$$s_1 = acfhbdeg, \quad s_2 = agdfbhce.$$

These generators are directly obtained from those given by Cole.*

From what precedes it is not difficult to deduce some fundamental properties of the infinite system of groups generated by two operators of order 8, each of which is transformed into its third power by the square of the other. In each of these groups $(s_1 s_2)^3$ generates an invariant subgroup whose operators are separately invariant under half the operators of the entire

*BULLETIN, vol. 2 (1893), p. 188.

group, while they are transformed into their inverses by the remaining operators. The fourth derived of each of these groups is identity while the third derived is of order 2. The quotient group of G with respect to the subgroup generated by $(s_1s_2)^3$, s_1^2, s_2^2, is the symmetric group of order 6 and this subgroup has the group of order 2 for its commutator subgroup. The quotient group of G with respect to the invariant group generated by $(s_1s_2)^3$ is either the symmetric group of order 24 or the group of 48 which may be generated by the two substitutions given in the preceding paragraph. Hence the theorem: *If two non-commutative operators are such that each is transformed into its third power by the square of the other, they are of the same order and this common order is* 2, 4, *or* 8. *If this order is* 2 *they generate a dihedral group; if it is* 4 *they generate a group whose quotient group with respect to a cyclic invariant subgroup is the symmetric group of order* 24; *and if it is* 8 *the corresponding quotient group is either this symmetric group or a group of order* 48 *which can be represented as a transitive substitution group of degree* 8.

If s_1, s_2 satisfy the two conditions

$$s_1^{2-}s_2s_1^{-2} = s_2^5, \quad s_2^{-2}s_1s_2^2 = s_1^5,$$

their orders divide 16, since $5^4 - 1$ is not divisible by 32. It also follows from the theorems of the preceding section that $s_1^8s_2^8 = 1$, and hence $s_1^8 = s_2^8$. To verify that the common order of s_1, s_2 may actually be 16 we may employ the following substitutions:

$$s_1 = abcdefghijklmnop, \quad s_2 = ahcfmlojipknedgb.$$

When s_1, s_2 represent these substitutions, H is of order 32 and involves a commutator subgroup of order 2. Since

$$s_1s_2 = bf \cdot cg \cdot em \cdot hp \cdot jn \cdot ko$$

the order of G is 128. In the next paragraph we shall prove that the order of H is 32 and that the order of G is a multiple of 128 whenever s_1 is of order 16.

The operator $(s_1s_2)^2$ is invariant under half the operators of G while it is transformed into its inverse by the remainder of these operators since

$$(s_1s_2)^{-2} = s_2^{-1}s_1^{-1}s_2^{-1}s_1^{-1} = s_2 \cdot s_2^{-2}s_1^{-1}s_2^2 \cdot s_2^{-3}s_1^{-1}$$
$$= s_2s_1^{-5}s_2^{-3}s_1^{-1} = s_2s_1 \cdot s_1^{-6}s_2^{-3}s_1^6 \cdot s_1^{-7} = (s_2s_1)^2.$$

As $(s_1s_2)^2$ is transformed into its inverse by s_1, it is also transformed into its inverse by s_2 and hence it is transformed into itself by half the operators of G. To prove that the order of H is exactly 32 when s_1 is of order 16 it is only necessary to observe that s_1^4 is not invariant under G, and this results from the continued equation

$$(s_1s_2)^{-1}s_1^4s_1s_2 = s_2^{-1}s_1^4s_2 = s_1^4 \cdot s_1^{-4}s_2^{-1}s_1^4 \cdot s_2 = s_1^4s_2^5 = s_1^{-4}.$$

These results give rise to the theorem: *If two operators of order* 16 *are such that each of them is transformed into its fifth power by the square of the other, these squares generate an invariant subgroup of order* 32 *involving a commutator subgroup of order* 2.

It has been proved that the group generated by $(s_1s_2)^2$ is invariant under G and that the order of the corresponding quotient group is a divisor of 128. To prove that this invariant subgroup has at most two operators in common with the invariant subgroup H it is only necessary to observe that a subgroup of order 4 in the former of these invariant subgroups can not be contained in the latter. This results directly from the facts that s_1s_2 is not commutative with s_1^4 or s_2^4 and that the operators of order 4 in H which are not generated by s_1^4 or s_2^4 are non-commutative with s_1^2, while $(s_1s_2)^2$ is commutative with this operator. Since the commutators of G are contained in the invariant subgroup generated by s_1^2, s_2^2, $(s_1s_2)^2$, and this invariant subgroup has a commutator subgroup of order 2, it results that the second derived of G is identity. It is also evident that the quotient group of G with respect to this invariant subgroup is the four-group. Hence the theorem: *If each of two operators is transformed into its fifth power by the square of the other, the orders of these operators divide* 16 *and the second derived of the group generated by them is identity.* Hence these operators may have a common order only when this order is 1, 2, 4, 8, or 16, and if they are non-commutative they can have different orders only when these orders are one of the two pairs 2, 4; 4, 8.

As a third and final special case we consider the relations

$$s_1^{-2}s_2s_1^2 = s_2^3, \quad s_2^{-2}s_1s_2^2 = s_1^5,$$

where α and β are unequal. According to the theorems of the preceding section, the orders of s_1, s_2 divide 8 and 4 respectively. It is easy to verify by means of substitutions that the orders of s_1, s_2 may actually be 8 and 4 respectively.

$$s_1 = abcdefgh \cdot ijlmnop, \quad s_2 = aiem \cdot cogk.$$

When s_1 is of order 8, s_2 must be of order 4 since s_2^2 is not commutative with s_1. When s_1 is of order 4, the order of s_2 is either 4, 2, or 1, as may be readily seen from the following substitutions:

$$s_1 = aecg, \quad s_2 = abcd \cdot efgh\,; \quad s_1 = abcd, \quad s_2 = ac.$$

Finally, when s_1 is of order 2, the order of s_2 is evidently 2 or 1. Hence the theorem: *If two non-commutative operators satisfy the relations* $s_1^{-2}s_2s_1^2 = s_2^3$, $\quad s_2^{-2}s_1s_2^2 = s_1^5$, *their orders are one of the following pairs of numbers:* 8, 4; 4, 4; 4, 2; 2, 2.

When s_1 is of order 8, H is abelian and of order 8. From the following equations it results that s_1^2 is transformed into its inverse by $(s_1s_2)^2$:

$$(s_1s_2)^{-1}s_1^2s_1s_2 = s_2^{-1}s_1^2s_2 = s_1^2 \cdot s_1^{-2}s_2^{-1}s_1^2 \cdot s_2 = s_1^2s^2,$$

$$(s_1s_2)^{-2}s_1^2(s_1s_2)^2 = s_2^{-1}s_1s_2^2s_1s_2 = s_2 \cdot s_2^{-2}s_1s_2^2 \cdot s_1s_2 = s_2s_1^{-2}s_2 = s_1^{-2}.$$

Hence the order of G is a multiple of $8 \cdot 4 \cdot 2 = 64$ whenever s_1 is of order 8. That the order of G may be exactly 64 results directly from the given substitutions, as they generate an imprimitive group of degree 16 and order 64. From the properties of the dihedral group it results that s_1, s_2 may be so selected that the order of G is an arbitrary multiple of 64 and that the third derived of each one of these groups is identity. The categories of groups which result when the orders of s_1, s_2 have the other possible sets of values are still more elementary and their fundamental properties are easily derived from the general theorems of the preceding section.

THE SOLUTION OF AN INTEGRAL EQUATION OCCURRING IN THE THEORY OF RADIATION.

BY PROFESSOR W. H. JACKSON.

(Read before the American Mathematical Society, December 30, 1909.)

PROFESSOR Arthur Schuster * has discussed the propagation of heat by radiation when the isothermal surfaces are

* "The influence of radiation on the transmission of heat." *Phil. Magazine*, Feb., 1903.

parallel planes and the radiation is homogeneous and everywhere normal to the isothermal surfaces.

If either of these two last restrictions is discarded, an equation is obtained of the form

$$E - \frac{1}{[k(\kappa)]^2}\frac{\partial^2 E}{\partial m^2} = \int_0^1 E d\kappa.$$

By a series of substitutions, the solution of this equation can be led back to the solution of an ordinary integral equation of the second kind with symmetric kernel. Each step is directly suggested by physical conceptions. Write

$$A = E - \frac{1}{k}\frac{\partial E}{\partial m}, \quad B = E + \frac{1}{k}\frac{\partial E}{\partial m},$$

whence

$$2E = A + B.$$

Put further

$$\theta(m) = \int_0^1 E d\kappa,$$

so that

$$A + \frac{1}{k}\frac{\partial A}{\partial m} = B - \frac{1}{k}\frac{\partial B}{\partial m} = \theta(m).$$

By the ordinary device of an integrating factor $e^{\pm km}$ these equations lead to

$$A = A_0(\kappa)e^{-km} + k\int_0^m e^{-k(m-\xi)}\theta(\xi)d\xi,$$

$$B = B_0(\kappa)e^{-k(m'-m)} + k\int_m^{m'} e^{-k(\xi-m)}\theta(\xi)d\xi,$$

where $0 < m < m'$ and A_0, B_0 are arbitrary functions of κ.

By addition, putting

$$2E_0 = A_0(\kappa)e^{-km} + B_0(\kappa)e^{-k(m'-m)},$$

we have

$$2E = 2E_0 + k\int_0^{m'} e^{-k|\xi-m|}\theta(\xi)d\xi.$$

Finally, integrating from 0 to 1 with respect to κ and writing

$$2K(x) = \int_0^1 ke^{-k|x|}d\kappa, \quad \theta_0(m) = \int_0^1 E_0 d\kappa,$$

we have

$$\theta(m) = \theta_0(m) + \int_0^{m'} K(\xi - m)\theta(\xi)d\xi.$$

The auxiliary function θ is the temperature and the obvious physical mode of solution is Liouville's method of successive substitutions.*

A case of special interest physically is that in which k is defined by

$$k\kappa = 1.$$

Is there any method of numerical computation better than approximate integration?

HAVERFORD COLLEGE,
March, 1910.

GRASSMANN'S PROJECTIVE GEOMETRY.

Projektive Geometrie der Ebene unter Benutzung der Punktrechnung dargestellt. Von HERMANN GRASSMANN. Erster Band: *Binäres.* B. G. Teubner, 1909. 8vo. xii + 360 pp.

MODERN projective geometry is two-sided. Either use is made of algebraic analysis in its development or it is developed from the fundamental concepts of point, line, plane by means of certain axioms and postulates. In the one case it is analytic, in the other synthetic. Usually the two methods of presentation are more or less combined, with the emphasis laid upon the one or the other. If the analytic method is adopted, operations are usually carried out in cartesian space with the aid of a system of coordinates. The synthetic method makes no use o coordinate systems.

Professor Grassmann's work is analytic in character in that use is made of algebraic analysis. It is unique in discarding the usual coordinate systems and adopting ideas due to Möbius and to the elder Grassmann. These ideas found expression in the Baryzentrische Calcul and in the Ausdehnungslehre.

In the last quarter century a number of writers have made use of these ideas; notably, Stéphanos, H. Wiener, Segre, Peano, Aschieri, Study, Burali-Forti. It is the author's purpose to bring the results of these writers and of others together into a connected course covering the fields of binary and ternary linear transformations. This is certainly a most worthy purpose and mathematicians will be grateful to the author for the evident care and devotion with which he has set about the performance of his task.

* Maxime Bôcher, An introduction to the study of integral equations, Cambridge, Eng., 1909.

The present volume is devoted to the binary field and is filled with detail much of which must be passed over without notice. It is furnished with a good table of contents and a register of material and names, which aid substantially in following the main lines of thought. What follows may serve to illustrate these. There are three Hauptteile or chapters. In the first of these the reader is introduced to the Punktrechnung by means of which points and lines are added, subtracted, and multiplied and the laws governing these operations formulated. Thus a point is represented by two factors, a scalar m called its mass and a vector f depending only upon its position. The sum of two points $m_1 f_1$ and $m_2 f_2$ is a point ms whose mass m is $m_1 + m_2$ and which has the position of the center of gravity of the two given points. Points of zero mass are ideal or at infinity. Points of unit mass are said to be simple points, others are called multiple points.

The process of multiplying one point a by another b is called exterior (äussere) multiplication and is indicated by the symbol $[ab]$. It is non-commutative, $[ab] = -[ba]$. The product is conceived of as a force acting along the segment from a to b and is called a Stab — a designation used by the author in 1894 (Punktrechnung und Projektive Geometrie, Halle) and since quite generally adopted by German writers. If a and b are finite points, the product $[ab]$ vanishes only when $a = b$.

The difference between two simple points is the segment (Strecke) connecting them. The point calculus will naturally be more familiar to the student of vector analysis than to one accustomed to think in terms of ordinary coordinate systems.

In the second chapter one finds an application of the point calculus to elementary projective geometry. A group of four points on a line in the order a, c, b, d, is called, after Von Staudt, a Punktwurf and its anharmonic ratio is defined as the double ratio

$$\frac{[ac]}{[cb]} \div \frac{[ad]}{[db]}.$$

This, being the double ratio of four Stäbe, on the same line, is a numeral quantity. The definition of projectively related ranges and pencils follows, together with the generation of curves of second class and of second order. The harmonic properties of the complete quadrilateral and the Pascal and Brianchon theorems are derived easily and thus we have the elementary part of projective geometry.

If the point calculus were an end in itself, or if it found its chief application in developing this elementary geometry, it would be rather more curious than useful. But the advantage comes in studying linear transformations in the n-ary field and is perhaps increasingly manifest the greater the value of n. Hence the reader's interest will be aroused and will increase as he reads on in the third and last chapter. This chapter is devoted to projectivities on a line and in a plane pencil and contains rather more than two thirds the entire volume.

For the analytic representation of the points of a point row, two non-coincident base points e_1 and e_2 are chosen whose masses are determined so that a third point e, not coinciding with either of the others, but otherwise arbitrary as to mass and position, shall be the sum of the other two. This third point is the unit point. Any point x of the point row is then given by the formula

$$x = x_1 e_1 + x_2 e_2,$$

where x_1 and x_2 are numerical quantities. A second point row on the same line whose base points and unit point are respectively a_1, a_2, and a is projectively related to the first when the base points and unit point of the one are made to correspond to the base points and unit point of the other, each to each.

So far this is not unlike the usual introduction to the study of linear transformations in the binary field. The divergence comes in the next step. A factor p* (Abbildungsfaktor) is defined so that

$$e_1 p = a_1,\ e_2 p = a_2,\ ep = a\,;\ i.\ e.,\ (e_1 + e_2)\, p = e_1 p + e_2 p,$$

and also

$$xp = (x_1 e_1 + x_2 e_2) p = x_1 a_1 + x_2 a_2.$$

The operation p so defined may be represented formally by what the author calls an extensive fraction

$$p = \frac{a_1,\ a_2}{e_1,\ e_2}.$$

This brings into evidence the base points of the two point rows. The operation indicated by p is distributive over the sum of any number of points on the line and associative when applied

* Many of the symbols in the text are expressed by German letters.

to the product of a point by a numerical quantity. Similar considerations hold for a projectivity established between two sheaves of lines belonging to the same plane pencil of rays.

The commutative and associative laws hold for the sum of any number of symbols representing projectivities on the same line or in the same pencil. The product of two such symbols p, q is called the resultant product (Folgeprodukt) and is defined as a third projectivity on the same line obeying the associative law

$$x(pq) = (xp)q.$$

The resultant product is in general non-commutative. The method for calculating this resultant product is given, and we are led at once to the theorem that all projectivities on a line form a group.

A projectivity $p = a_1,\ a_2/e_1,\ e_2$ possesses an inverse $p^{-1} = e_1,\ e_2/a_1,\ a_2$, provided the exterior product $[a_1 a_2] \neq 0$; *i. e.*, provided the base points of the second point row are not coincident. In the opposite case, p is degenerate and does not possess an inverse. If p possess an inverse, the equation

$$xp = x'$$

is solvable for x.

The combination product $[yz \cdot pq]$, where y and z are points on the line, is defined by the equation

$$[yz \cdot pq] = \tfrac{1}{2}\{[yp \cdot zq] - [zp \cdot yq]\}.$$

A number multiplying any one of the letters y, z, p, q may be written before the entire expression, and the symbol itself is distributive over a sum replacing any one of its constituents. The expression

$$\frac{[yz \cdot pq]}{[yz]}$$

is independent of the particular points y, z used in forming it. Hence

$$\frac{[yz \cdot pq]}{[yz]} = \frac{[e_1 e_2 \cdot pq]}{[e_1 e_2]} \equiv [pq].$$

This new symbol $[pq]$ is called the combination product of the projectivities p and q, and is always commutative in contradistinction to the resultant product.

The combination square or power of a projectivity

$$[p^2] = \frac{[e_1e_2 \cdot pp]}{[e_1e_2]} = \frac{[a_1a_2]}{[e_1e_2]}$$

is a numerical quantity, since it is the ratio of the two Stäbe connecting the base points of each point row.

A projectivity p is directly or oppositely projective; i. e., the points of the two point rows are arranged in the same or in opposite senses according as $[p^2]$ is positive or negative. If $[p^2] = 0$, then $[a_1a_2] = 0$ and the projectivity is degenerate.

A double point d of a projectivity p satisfies the equation

$$dp = rd,$$

where r is a numerical quantity, hence it is altered only as to mass by the projectivity. This equation easily transforms into a quadratic in r whose coefficients depend only upon the base points of the two point rows. The roots of this equation are termed the principal numbers of the projectivity and the equation itself its principal equation.

If a projectivity leaves every point of the line unaltered as to mass and position, it is identity. If it alters the mass only, it is a coincidence (Deckung). If the sum of the principal numbers is zero, the projectivity is an involution. The properties of the involution follow together with an application to the vector equations of the ellipse and the hyperbola. The equation of the ellipse, for example, comes out to be

$$x = a(\cos w + \mathbf{e} \sin w) = ae^{\mathbf{e}w},$$

where x is any point of the curve, a the semi-major axis, w the eccentric angle, e the Naperian base, and $\mathbf{e}$ the elliptic involution which changes one set of diameters into the conjugate set.

The consideration of projectivities with conjugate imaginary principal numbers is introduced by a study of what are called positive and negative circular forms (Abbildungen), after a designation due to the elder Grassmann. A positive circular form in the plane pencil is a projectivity transforming a sheaf of lines into a congruent sheaf, i. e., a rotation. A positive circular form upon a line is a section of the congruent sheaves by a line not passing through their common center. Every projectivity with conjugate imaginary principal numbers is identical geometrically to a positive circular form and can differ from it analytically only by a numerical factor.

A negative circular form in the plane pencil transforms a sheaf into a congruent sheaf, the rays of the two sheaves being arranged in opposite senses; i. e., it is a reflection across one of the double rays of the projectivity. A section by a line not passing through the common center gives rise to a negative circular form upon that line.

A circular form is expressible exponentially in terms of a certain involution. For example, the positive circular form on a line is

$$C_{(a,\,b,\,w)} = e^{\mathfrak{e}\,w},$$

where a, b are the base points of the first point row, w is a real parameter, e the Naperian base, and $\mathfrak{e}$ is the elliptic involution

$$\mathfrak{e} = \frac{b,\ -a}{a,\ b}.$$

For a negative form, the involution in the exponent is hyperbolic

$$h = \frac{b,\ a}{a,\ b}.$$

The double points of these involutions coincide with the double points of the corresponding circular form.

All positive circular forms with the same point pair a, b form a continuous one-parameter group. Negative circular forms do not form a group. The totality of all positive and negative circular forms with the same point pair a, b forms a discontinuous group which contains the continuous group of positive circular forms as a subgroup.

The circular forms are not the only projectivities expressible exponentially in terms of an involution. The consideration of linear systems of projectivities, in particular a sheaf

$$lp + mq,$$

where l and m are numerical quantities, leads to the fact that, unless all the projectivities of the sheaf are involutions, there is but one involution contained in the sheaf. If the projectivity q is fixed, say identity, the double points of the involution contained in the sheaf coincide with the double points of p, and p is expressible in terms of the identity and this involution double points.

A direct (gleichläufig) projectivity with real and distinct double points is then

$$p = e^{hw},$$

where h is the double point involution and w a real parameter. On the other hand an opposite (gegenläufig) projectivity expressed in terms of its double points involution is

$$q = he^{hw}.$$

Group properties follow as in the case of the circular forms.

Two projectivities for which the combination product $[pq]$ vanishes are called harmonic. The name is due to Segre as well as many of their properties (*Crelle*, volume 100).

The domain of all the projectivities on a line is defined by means of four unit fractions (Ausdehnungslehre)

$$\mathbf{e}_{11} = \frac{e_1, 0}{e_1, e_2}, \quad \mathbf{e}_{12} = \frac{e_2, 0}{e_1, e_2}, \quad \mathbf{e}_{21} = \frac{0, e_1}{e_1, e_2}, \quad \mathbf{e}_{22} = \frac{0, e_2}{e_1, e_2}.$$

These are all degenerate projectivities, the second and third being parabolic involutions, and hence the power of each vanishes. Any projectivity

$$p = \frac{a_1, a_2}{e_1, e_2}$$

can be expressed as a linear function of these four units. For if we put

$$a_1 = a_{11}e_1 + a_{12}e_2, \quad a_2 = a_{21}e_1 + a_{22}e_2,$$

then

$$p = a_{11}\mathbf{e}_{11} + a_{12}\mathbf{e}_{12} + a_{21}\mathbf{e}_{21} + a_{22}\mathbf{e}_{22}.$$

The proof consists in applying p, thus expressed, to each of the base points e_1, e_2 and showing that they transform into a_1, a_2 respectively.

The four units are linearly independent. For the assumption of an equation of the form

$$c_{11}\mathbf{e}_{11} + c_{12}\mathbf{e}_{12} + c_{21}\mathbf{e}_{21} + c_{22}\mathbf{e}_{22} = 0$$

leads to the vanishing of all the c's since e_1 and e_2 are distinct points.

The above representation of p is unique. For if p could be expressed by a set of a's and also by a set of b's, the difference between the two expressions would vanish and thus the a's equal the b's, each to each.

Of course any four linearly independent projectivities may be taken for a base. It is convenient to choose four which are mutually harmonic; these are, when expressed in terms of the unit fractions,

$$1 = e_{11} + e_{22}, \quad h_1 = e_{11} - e_{22}, \quad h_2 = e_{12} + e_{21}, \quad e = e_{12} - e_{21},$$

where 1 is identity, h_1 and h_2 are hyberbolic involutions, and e is an elliptic involution. It is easily shown that they are mutually harmonic by means of the laws governing the multiplication of unit fractions and the condition which two projectivities must satisfy in order to be harmonic. Any projectivity on the line is then

$$p = a + a_1 h_1 + a_2 h_2 + a_3 e.$$

This expression of p leads at once to the Stéphanos (*Mathematische Annalen*, volume 22) representation of the projectivities on a line by the points of ordinary space; viz., the images of the four fundamental projectivities 1, h_1, h_2, e are taken to be the vertices of a fundamental tetrahedron whose unit point is arbitrary. The image of any projectivity p is then the point whose homogeneous coordinates referred to this tetrahedron are a, a_1, a_2, a_3.

The images of all degenerate projectivities fill the quadric

$$a^2 - a_1^2 - a_2^2 + a_3^2 = 0$$

with respect to which the fundamental tetrahedron is self-conjugate. Two harmonic projectivities are imaged upon the points which are conjugate with respect to this quadric. The involutions on the line are imaged upon the points of the plane determined by the images of the three fundamental involutions h_1, h_2, and e.

A number of details of this representation are given together with a discussion of the representation of the involutions on the line by the points of a plane.

The projectivities upon a line form a system of higher complex quantities since they satisfy the four conditions laid down by Study (*Göttinger Nachrichten*, 1889). On comparing the multiplication table of the four fundamental projectivities with the multiplication table of the four units $1, i, j, k$ of the Hamilton quaternion system, it follows that any projectivity may be expressed as a complex quaternion, viz.,

$$p = a + a_1\sqrt{-1}i + a_2\sqrt{-1}j - a_3 k.$$

A theorem called Study's theorem (Cf. Encyclopädie, I. 1) brings these results together:

The group of all the projectivities on a line forms a system of

higher complex quantities of quaternion type. It possesses another real form which is like the system of ordinary quaternions and these two real forms are the only ones in which systems of higher complex quantities of quaternion type can appear.

The work is manifestly a labor of love. An interesting circumstance in connection is the fact that this first volume was brought out on the hundredth anniversary of the birthday of the author's father.

L. WAYLAND DOWLING.

LINEAR DIFFERENTIAL EQUATIONS.

Vorlesungen über lineare Differentialgleichungen. Von LUDWIG SCHLESINGER. Leipzig and Berlin, Teubner, 1908.

IT is over ten years ago that the author of the present "Vorlesungen" completed the publication of his well-known Handbuch der Theorie der linearen Differentialgleichungen. As every one familiar with the older book well knows, it was intended to be, as its name implied, a handbook containing a complete treatment of all that was at that time known about the subject. It seemed natural therefore, to expect under the title of "Vorlesungen" a briefer version of the same subject, adapted to the needs of the younger student and rendered more palatable for him by a proper selection of topics and by a more elementary treatment. And in a certain sense the "Vorlesungen" may indeed be considered as an introduction into the theory of linear differential equations, in so far at least as all of the most important results of the theory built up by Fuchs and his successors are discussed. But the method of treatment is so novel and the artistic unity of the book is preserved to such an extraordinary extent that we must look upon it as an important addition to analysis rather than as a treatise of more or less pedagogical merit.

It is well known that Riemann's discussion of the hypergeometric function furnished Fuchs with the fundamental ideas which led to the modern theory of linear differential equations, which theory may be said to date from Fuchs's paper of 1865. But we now know that Riemann himself had intended to construct a general theory of linear differential equations upon the same general principles which had led to such brilliant results in

the theory of abelian functions. In the two fragments, published in 1876, after his death, he formulates a general problem which may briefly be stated as follows: Given m points $a_1, \cdots, a_m$ in the plane of the complex variable with each of which is associated one of the linear n-ary substitutions with constant coefficients $A_1, \cdots, A_m$; to determine a system of n functions of the complex variable x which shall have the given points $a_1, \cdots, a_m$, and no others, as branch points in such a way that when x describes a closed path in the positive direction around a_i, the functions y_κ shall undergo the linear substitution A_i. Riemann speaks of two different systems of functions which are nowhere "infinite of infinite order" which have the same branch points, the same fundamental substitutions and the same poles, as belonging to the same *class*.

It is easy enough to see that the solutions of a linear differential equation of the nth order are functions of this general character. But if the points a_i and the substitutions A_i and the poles are chosen arbitrarily, it is a problem of great difficulty to demonstrate the existence of a system of functions of the class defined by them. This is what is known as Riemann's problem and in its solution Schlesinger's Vorlesungen culminate. The method adopted consists essentially in studying the relations between the parameters which occur in the coefficients of the differential equation, the quantities which determine the linear substitutions A_i and the branch points a_i. These (transcendental) relations are shown to be of such a character that, the latter quantities being arbitrarily assigned, the former may be chosen in such a way that the corresponding differential equations have as their solutions a system of functions with the required properties.

As we have already indicated, the solution of Riemann's problem is the culminating feature of the book and appears only at the end of a long series of investigations in which all of the most essential properties of linear differential equations are studied. But all of these are presented in a novel way. Instead of considering the problems of integration in connection with a single linear differential equation of the nth order, Schlesinger follows Koenigsberger's example by considering a system of n differential equations of the first order. Such a system is characterized by its n^2 coefficients as well as by a system of n^2 solutions. It is owing to this fact that the entire theory appears as an application of the calculus of matrices of

n^2 functions which was originally developed by Volterra, and which certainly manifests itself here as a most important analytical instrument.

Let us consider the system of differential equations

$$(B) \qquad \frac{dy_\kappa}{dx} = \sum_{\lambda=1}^{n} y_\lambda a_{\lambda\kappa}(x) \qquad (\kappa = 1, 2, \cdots, n),$$

where the coefficients $a_{\lambda\kappa}$ are supposed to be real, finite and continuous functions of x in the interval $(p \cdots q)$. Let the values

$$(I) \qquad y_1 = y_1^{(0)}, \quad \cdots, \quad y_n = y_n^{(0)}$$

be arbitrarily prescribed for $x = x_0 = p$. In order to demonstrate the existence of a system of real continuous solutions of (B) which satisfy these initial conditions, the author makes use of the method of interpolation, which proceeds as follows, in close analogy with Riemann's definition of a definite integral: Let r be any value of x for which

$$p < r \leqq q.$$

Divide the interval $(p \cdots r)$ into m parts by interpolating $m-1$ points $x_1, \cdots, x_{m-1}$, and let us put $x_0 = p$, $x_m = r$. In each of the m subintervals $(x_{\nu-1} \cdots x_\nu)$ obtained in this way choose an arbitrary value $\xi_{\nu-1}$,

$$x_{\nu-1} \leqq \xi_{\nu-1} < x_\nu,$$

and consider in place of the differential equations (B), the difference equations

$$y_\kappa^{(1)} - y_\kappa^{(0)} = (x_1 - x_0) \sum_{\lambda=1}^{n} y_\lambda^{(0)} a_{\lambda\kappa}(\xi_0),$$

$$(C) \qquad y_\kappa^{(2)} - y_\kappa^{(1)} = (x_2 - x_1) \sum_{\lambda=1}^{n} y_\lambda^{(1)} a_{\lambda\kappa}(\xi_1), \quad (\kappa = 1, 2, \cdots, n),$$

$$\cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots$$

$$y_\kappa^{(m)} - y_\kappa^{(m-1)} = (x_m - x_{m-1}) \sum_{\lambda=1}^{n} y_\lambda^{(m-1)} a_{\lambda\kappa}(\xi_{m-1}).$$

The author shows in the first lecture that, under the assumptions made in regard to the continuity of the functions $a_{\lambda\kappa}$, the functions $y_\kappa^{(m)}$ defined by these difference equations approach definite limits y_κ as m becomes infinite, while each of the subintervals approaches the limit zero, these limits being indepen-

dent of the particular method of subdivision employed, as well as of the choice made of the points $\xi_{\nu-1}$ in the various subintervals. Moreover the n functions obtained in this way are shown to be continuous functions of x which satisfy the system of differential equations (B) and the initial conditions (I).

In the second lecture we are told to consider not a single system of solutions (B), but n such systems corresponding to the n systems of initial conditions

$$\begin{aligned} y_1 &= y_{11}^{(0)},\ y_{21}^{(0)},\ \cdots,\ y_{n1}^{(0)}, \\ &\cdots\cdots\cdots\cdots\cdots\cdots \qquad \text{for } x = x_0 = p, \\ y_n &= y_{1n}^{(0)},\ y_{2n}^{(0)},\ \cdots,\ y_{nn}^{(0)}, \end{aligned}$$

where the determinant

$$|y_{i\kappa}^{(0)}|$$

is supposed to be different from zero. The n systems of solutions $(y_{11}, \cdots, y_{1n}), \cdots, (y_{n1}, \cdots, y_{nn})$ will be obtained by the limit process just indicated from the difference equations

$$(C') \qquad y_{i\kappa}^{(\nu)} - y_{i\kappa}^{(\nu-1)} = (x_\nu - x_{\nu-1}) \sum_{\lambda=1}^{n} y_{i\lambda}^{(\nu-1)} a_{\lambda\kappa}(\xi_{\nu-1}),$$
$$(i, \kappa = 1, 2, \cdots, n;\ \nu = 1, 2, \cdots, m)$$

which are merely the equations (C) written down for each separate set of initial conditions.

The equations (C') may be written in the form

$$(1) \qquad y_{i\kappa}^{(\nu)} = \sum_{\lambda=1}^{n} y_{i\kappa}^{(\nu-1)} [a_{\lambda\kappa}(\xi_{\nu-1})(x_\nu - x_{\nu-1}) + \delta_{\lambda\kappa}],$$

where $\delta_{ii} = 1, (i = 1, 2, \ldots n)$, and $\delta_{i\kappa} = 0$ for $i \neq \kappa$. But these may be looked upon as the equations for the multiplication of two square matrices, since the single relation between matrices

$$(a_{i\kappa})(b_{i\kappa}) = (c_{i\kappa})$$

is equivalent to the n^2 relations

$$c_{i\kappa} = \sum_{\lambda=1}^{n} a_{i\lambda}\, b_{\lambda\kappa}, \qquad (i, \kappa = 1, 2, \ldots n)$$

and since equations (1) are obviously of this form.

Let us follow the author in using the simple parenthesis as a symbol for a matrix as well as in its elementary significance, although that practice occasionally gives rise to a formidable

collection of parentheses in a single equation. We suggest incidentally that this might easily be avoided, in the interests of clearness, by using some other symbol for a matrix, say a square bracket. Equations (1) may then be written as follows:

$$(2)\quad (y_{i\kappa}^{(\nu)}) = (y_{i\kappa}^{(\nu-1)})(a_{i\kappa}(\xi_{\nu-1})\cdot(x_\nu - x_{\nu-1}) + \delta_{i\kappa}) \quad (\nu = 1, \ \ldots, m),$$

which gives rise to the symbolic formula

$$(3)\qquad (y_{i\kappa}^{(m)}) = (y_{i\kappa}^{(0)})\prod_{\nu=1}^{m}(a_{i\kappa}(\xi_{\nu-1})\cdot(x_\nu - x_{\nu-1}) + \delta_{i\kappa}),$$

where the product upon the right member is, of course, a product of matrices taken in the proper order.

Let the matrix $(y_{i\kappa}^{(0)})$ of initial values be the unit matrix $(\delta_{i\kappa})$. The author now introduces the following symbol:

$$(\alpha)\qquad \overbrace{\int_p^r}[a_{i\kappa}(x)dx + \delta_{i\kappa}]$$

to denote the matrix of the limits

$$\lim_{m=\infty}\prod_{\nu=1}^{m}[a_{i\kappa}(\xi_{\nu-1})\cdot(x_\nu - x_{\nu-1}) + \delta_{i\kappa}] \quad (i, \kappa = 1, 2, \cdots, n).$$

The symbol (α) is to be read "integral matrix from p to r," and is used with great success throughout the book. The author tells us that he has chosen this symbol so as to emphasize its analogy with the ordinary integral, and at the same time to remind us of the initial letter of the word product, just as the integral sign reminds us of the first letter of the word sum.

Let us put $r = x$ and think of x as variable in the interval from p to q. The integral matrix

$$[\eta_{i\kappa}(x)] = \overbrace{\int_p^r}[(a_{i\kappa}(x)dx + \delta_{i\kappa}]$$

will represent a matrix of n^2 functions such that

$$\eta_{i1}(x), \ \cdots, \ \eta_{in}(x) \qquad (i = 1, 2, \cdots, n)$$

constitute n systems of simultaneous solutions of system (B) which satisfy the initial conditions

$$\eta_{i\kappa}(p) = \delta_{i\kappa},$$

and the most general matrix of simultaneous solutions is obtained from $(\eta_{i\kappa})$ in the form

$$(y_{i\kappa}) = (c_{i\kappa})(\eta_{i\kappa}),$$

where the elements of the matrix $(c_{i\kappa})$ are arbitrary constants.

The equations (B) may be written

$$\left(\frac{dy_{i\kappa}}{dx}\right) = (y_{i\kappa})(a_{i\kappa}),$$

whence

$$(a_{i\kappa}) = (y_{i\kappa})^{-1}\left(\frac{dy_{i\kappa}}{dx}\right),$$

the determinant of the matrix $(y_{i\kappa})$ being different from zero in the whole interval $(p \cdots q)$. Thus the coefficients of (B) may be expressed in terms of n systems of simultaneous solutions. Schlesinger denotes this operation by the symbol $D_x(y_{i\kappa})$, so that

$$D_x(y_{i\kappa}) = (y_{i\kappa})^{-1}\left(\frac{dy_{i\kappa}}{dy}\right),$$

and speaks of this as the derivative matrix of $(y_{i\kappa})$ with respect to x. If the solutions of (B) are not subjected to the initial conditions

$$\eta_{i\kappa}(p) = \delta_{i\kappa}$$

nor any other specific conditions, he writes

$$(y_{i\kappa}) = \widehat{\int}(a_{i\kappa}dx + \delta_{i\kappa}),$$

a notation which corresponds to the indefinite integral of ordinary analysis.

At the end of the second lecture we find the laws of combination of these new symbols of derivation and integration, which correspond closely to the familiar ones of the infinitesimal calculus, the principal difference being that symbolic multiplication of matrices takes the place of addition.

The third lecture introduces the integrating factors of Lagrange and Jacobi, and the idea of adjoint system, which is used in the familiar way for the purpose of showing how to integrate non-homogeneous systems by quadratures if the corresponding homogeneous systems have already been solved. This theory again finds an important application in enabling us to deduce, by successive approximation, a series for the solu-

tions of the system (B), which is convergent in the whole domain of continuity of the coefficients.

The transition from functions of a real to functions of a complex variable is made by following the same general ideas that dominate the corresponding step in the theory of ordinary integrals. By separating all of the variables involved into their real and imaginary parts, i. e., by putting

$$y_\kappa = u_\kappa + \sqrt{-1}v_\kappa, \quad a_{i\kappa} = \alpha_{i\kappa} + \sqrt{-1}\beta_{i\kappa}, \quad x = \xi + \sqrt{-1}\eta,$$

a system of total linear differential equations is obtained from (B). Its integral matrix is shown to be independent of the path, if this path be restricted to a simply connected portion of the plane at every point of which the coefficients $a_{i\kappa}$ are analytic. It then follows easily that all of the values of an integral matrix for a given value of x, taken over a path not so restricted, may be obtained from one of them by multiplication with constant matrices. This leads to the notion of fundamental equation, the representation of the elements of a canonical fundamental system in the well-known form and all of the rest of the classical theory.

Schlesinger's solution of Riemann's problem has had to undergo some severe criticisms by Plemelj.* The discussion initiated by this attack is certainly worthy of careful attention on the part of all who might be tempted to apply these methods to similar problems. The reviewer has not been able to find the time to form a final opinion upon the main points involved in this controversy. It certainly does not invalidate the value of the work as a whole.

The following unimportant misprints may be noted: page 10, line 23, read $\underset{\lambda-2}{r}$ in the exponent in place of $\underset{\lambda-1}{r}$; page 40, line 13, read $w_{i\kappa}^{(\nu)}$ instead of $(w_{i\kappa}^{(\nu)})$; page 76, equation (12), read $a_{i\kappa}$ instead of $\alpha_{i\kappa}$.

E. J. WILCZYNSKI.

*See *Jahresbericht der Deutschen Mathematiker-Vereinigung*, January 31, 1909, and June 11, 1909.

SHORTER NOTICES.

La Geometria non-euclidea, esposizione storico-critica del suo Sviluppo. Da ROBERTO BONOLA. Bologna, Zanichelli, 1906. vi + 213 pp.

Wissenschaft und Hypothese IV: *Die nichteuklidische Geometrie, historisch-kritische Darstellung ihrer Entwicklung.* Von ROBERTO BONOLA, Professor an der Scuola Normale zu Pavia. Autorisierte deutsche Ausgabe besorgt von Prof. Dr. HEINRICH LIEBMANN. Leipzig, Teubner, 1908. viii + 244 pp.

IN the development of the sober science of mathematics a certain dramatic and even sensational element has been furnished by non-euclidean geometry, the history of which is therefore unusually interesting. By its very nature this subject lends itself easily to a historical and critical treatment, like that of the admirable book under review. Unfortunately, the very fascination of the subject has apparently retarded its growth along the substantial lines of actual detailed knowledge: the tendency has been to regard it as a curious and elegant plaything, rather than as the valuable adjunct to euclidean geometry, which it undoubtedly is. The slowness of its growth is illustrated by the fact that although more than eighty years have elapsed since Lobachevsky published his first epoch-making researches, it is only recently that quadric surfaces in non-euclidean space have been carefully studied and classified.

This and most other recent investigations are not mentioned by Bonola. Indeed, as he himself states, the character of his book is distinctly elementary. He begins by taking the reader back to the early period of questioning and doubt as to Euclid's fifth postulate, then carries him through the storm and stress period of creation by Gauss, Lobachevsky, and Bolyai, and is finally content to land him safely in the harbor of modern thought, where projective geometry, differential geometry, and the theory of continuous groups all afford cumulative evidence of the validity of the new doctrine.

In Chapter 5 the author gives several well-known methods of representing or imaging a non-euclidean space in a euclidean space, but omits to mention one introduced by Klein and oincaré and used successfully by Weber and Wellstein in their Encyklopädie der Elementar-Geometrie and by Liebmann (the

translator of Bonola's work into German) in his own Nichteuklidische Geometrie, namely, one in which non-euclidean lines and planes are represented by euclidean circles and spheres, respectively.

Two supplementary chapters, one on non-euclidean statics and one on Clifford parallels, Clifford surfaces, and the Clifford-Klein problem, and in the German edition another supplementary chapter on the construction of Lobachevsky parallels, add considerably to the value of the book. There is an index of authors cited, but no general index.

It would be fortunate if we could have an English translation of so valuable and interesting a work; for in English there is nothing covering even approximately the same ground except possibly the scattered papers of G. B. Halsted.

ARTHUR RANUM.

Analytic Geometry. Revised Edition. By E. W. NICHOLS. D. C. Heath and Company, 1908. xi + 282 pp.

THE general scope of this book is the same as that of the first edition which appeared fifteen years earlier. In the first edition the last chapter—a discussion of surfaces—was written by Professor A. L. Nelson and in the new edition this chapter has been entirely rewritten. Otherwise comparatively few changes in the subject matter have been made. The revised edition is very neatly bound in flexible covers—the style so largely used by D. C. Heath and Company lately. The printing, too, is distinctly better than in the former edition.

"The aim of the author has been to prepare a work for beginners, and at the same time to make it sufficiently comprehensive for the requirements of the usual undergraduate course." The first part of this aim has been more successfully carried out than the second. The book is written clearly and contains numerous, well-chosen problems. The conventional order of topics is followed—the conic sections being discussed separately with little emphasis upon their relation to each other. Probably the book is more elementary than would be acceptable in the best engineering schools.

G. H. SCOTT.

Complete Arithmetic. By GEORGE WENTWORTH and DAVID EUGENE SMITH. Boston, Ginn and Company, 1909. v + 474 pp.

THIS book preserves and combines most of the strong features of two well-known series of arithmetics—the Wentworth and

the more modern Smith texts. Briefly, it embodies the spirit of the newer series in the forms of the older. It is "thoroughly modern in spirit and in material" but is free from all traces of "fad-ism" found in so many texts of recent years. The book is strictly topical, although the authors frankly admit, in the preface, that under certain conditions the recurrent treatment of topics may be preferable. The problem material is carefully chosen and is given in great abundance. And, at frequent intervals, under the heading "Problems without Numbers" are given sets of questions that combine review and generalization very effectively. It might have been well to present the metric system earlier and then give practical problems in it through a longer interval.

G. H. SCOTT.

Theorie des Potentials und der Kugelfunktionen. Von DR. A. WANGERIN. I. Band., Leipzig, Göschen (Sammlung Schubert. Band LVIII.), 1909. 8 + 255 pp. M. 6.60.

THIS is the first of two volumes dealing in an elementary way with the subject indicated by the title. The second volume will treat of spherical harmonics and their applications to the potential of the sphere. The present volume is confined to the derivation of the characteristic properties of the potential. The treatment follows Gauss for the potential due to solids, Weingarten for that due to surfaces. The potential function for other laws than the Newtonian is briefly considered. The last section gives in some detail the problems of potential and attraction of a homogeneous ellipsoid.

The development is very skilfully handled. The text begins with very elementary data, and builds up the integrals for the attractions of solids and surfaces, with applications to circular arcs, straight segments, and surface of circle and sphere. It is thus made to connect easily with an ordinary course in integral calculus. The potential function noticed by Lagrange is then introduced as a point function whose three partial derivatives are the three components of the attraction. The conceptions of equipotential surface and lines of force follow. The next chapter derives the usual characteristic properties of the potential function, as a function of a position in space, for points outside the attracting mass. The holomorphism of the function and its derivative as to x, y, or z, its order at infinity, and the vanishing of its concentration are shown. Next the characteristics for

points belonging to the attracting mass are developed, and the discontinuity of the second derivatives as the point goes through a bounding surface is shown and the values of the saltus determined. Following this, like problems are taken up for surface distributions. The closing chapter of the section is devoted to proving that the properties enumerated for the potential as necessary are also sufficient, hence characteristic.

The second section considers the function for other laws than that of the inverse square of the distance. It is shown in particular that the Newtonian is the only law which gives a constant potential inside a spherical shell whose density is a function of the distance from the center. It is not however the only law for which the attraction on an external point due to the shell is equal to that of an equal mass concentrated at the center of the shell. The shape of the "solid of greatest attraction" is considered. The logarithmic potential and the potential due to a double distribution, as a Leyden jar, are each given a chapter.

The book would seem to be quite teachable. Gauss's, Green's, and Stokes's theorems are not dragged in, but show up naturally when needed to further the development. The student sees clearly all the time the drift of the development and why it proceeds as it is does. He learns how to attack such problems, but he also becomes acquainted with a class of point functions particularly useful in mathematical physics. Difficult questions of higher analysis are passed over, yet the treatment is careful and tends to inspire to further research.

JAMES BYRNIE SHAW.

Geometrie der Kräfte. By H. E. TIMERDING. Leipzig and Berlin, Teubner, 1908. 8vo. xi + 381 pages.

In this book the author has developed the geometry of forces as an independent discipline, a branch of pure mathematics. While the word force (Kraft) has been retained in preference to stroke or vector, great pains have been taken to free it from the "physiological, physical, and metaphysical" attributes which belong to it originally. A force is a matter of definition, being defined as a vector with which is associated a numerical factor. The resulting theory is then applicable to any quantity which satisfies the definition, for example to momentum quite as well as to force in the ordinary physical sense. The subject matter is not new. In different forms and

from various points of view it has appeared in several places. For example the part which is applicable to the kinetics of rigid bodies has been worked out in Ball's Theory of Screws, and a parallel development for the statics of deformable bodies appears in the writings of Sir William Thomson. The authors' purpose in rewriting to a large extent the Theory of Screws is explained by the following quotation from the preface. "As a consequence of the English genius it (Ball's work) shows, even in those parts which do not have immediate practical applications, a fine appreciation for the essence and claims of science. On the other hand it seeks in no way to attain to that unity and purity which we Germans have been accustomed to regard as the goal of all scientific work. It appears more as a geometric illustration of mechanics than as a special subject independent of mechanics. The latter appears to us as the only purpose which a geometry of forces has to fulfill." It is intended that this development shall provide a bridge from geometry to mechanics without assuming a hybrid character; that it shall be a development such as the geometry of motion has had under the name of kinematics.

The first five chapters stand apart from the rest of the book and are devoted to an exposition of Grassmann's Ausdehnungslehre and Hamilton's Quaternions. Any mention of vector products carries with it the question as to whose notation is employed. After giving in a footnote the notations of Grassmann, Hamilton, Heaviside, Gibbs, and Lorentz, Professor Timerding adopts one which is not in agreement with any of the others. He denotes the inner or scalar product by $\mathfrak{a} \times \mathfrak{b}$ and the outer or vector product by $[\mathfrak{ab}]$. In the following chapters almost no use is made of the terminology and results of vector analysis, and much of the first five chapters could have been omitted. It is remarked in the preface that the introduction of the methods of Grassmann's analysis into the first part of the book without making use of them in the sequel may be open to criticism. The justification is to be found in the purpose of a strictly systematic development of the subject without regard to subsequent application of the early steps.

For the principal part of the book the choice of material has been determined by two main considerations. On the one hand a development of the geometry of forces must be broad enough to include the two manifestations of mechanical energy, the kinetic energy of motion and the potential energy of deforma-

tion. On the other hand the author has demanded a minimum prerequisite knowledge of analysis. This requirement is limited to elementary analytic geometry and calculus, to which may be added a few of the formulas for vector products given in the introductory chapters.

The exposition of the geometry of forces begins in Chapter 6 with a consideration of instantaneous rotations. This is followed by chapters on forces and force systems, foundations of line geometry, and equilibrium. The next six chapters are devoted to the theory of screws and are followed by two chapters on deformations, the point of view throughout this portion being purely geometric. The concepts of mechanics are taken up in the last six chapters, of which two are given to deformable bodies. The four chapters on kinetics of rigid bodies deal with the equations of motion in general, free motion under no applied forces, motion with two degrees of freedom, and with three degrees of freedom. The special case of a system of forces in a plane is excluded throughout, and in the free motion of a rigid body it is assumed that the axis of rotation does not have a fixed direction in space.

The necessary complications of notation in this subject have been reduced by the systematic use of different styles of type, thus avoiding an excessive number of accents and subscripts. For the convenience of the reader the scheme of notation is exhibited in a table at the end of the book.

There are many misprints, but fortunately most of them are self-evident and will not cause confusion.

W. R. LONGLEY.

NOTES.

THE seventeenth summer meeting of the AMERICAN MATHEMATICAL SOCIETY will be held at Columbia University on Tuesday and Wednesday, September 6–7. Abstracts of papers intended for presentation at this meeting should be in the hands of the Secretary not later than August 20.

THE April number (volume 32, number 2) of the *American Journal of Mathematics* contains the following papers: "The reduction of families of bilinear forms," by H. E. HAWKES; "Basic systems of rational norm-curves," by J. R. CONNER; "Surfaces invariant under infinite discontinuous birational

groups defined by line congruences," by V. SNYDER; "The apparent size of a closed curve," by C. A. LUNN; "On linear transformations which leave an Hermitian form invariant," by J. I. HUTCHINSON.

THE April number (volume 11, number 3) of the *Annals of Mathematics* contains: "A generalization of the game called nim," by E. H. MOORE; "A simple method for graphically obtaining the complex roots of a cubic equation," by R. E. GLEASON; "The topography of certain curves defined by a differential equation," by F. R. SHARPE; "Abel's theorem and some addition formulæ for elliptic integrals," by H. H. BARNUM; "On the determination of the asymptotic developments of a given function," by W. B. FORD; "The integral roots of certain inequalities," by W. H. JACKSON.

AT the meeting of the London mathematical society held on April 28 the following papers were read: By W. F. SHEPPARD, "On the accuracy of interpolation by finite differences;" by G. H. HARDY, "Note on Maclaurin's test for the convergence of series;" by A. J. C. CUNNINGHAM, "The factorization of $2^{77}+1$ and the divisibility of 2^p-2 by p^2, p being prime."

THE proceedings of the Fourth international congress of mathematicians, which was held at Rome in April, 1908, have been distributed to the members, and can now be obtained from the publisher. Especially should public and university libraries avail themselves of the opportunity to procure this carefully prepared digest of the progress of pure and applied mathematics. The complete report appears in three volumes, and can be obtained from the firm of E. Loescher, in Rome, for 35 francs. The volumes are not sold separately. The titles are as follows: Atti del IV Congresso internazionale dei Matematici, publicati per cura del Segretario Generale G. CASTELNUOVO, Roma; Tipografia della R. Accademia dei Lincei. Volume I (Relazione sul congresso, Discorsi e conferenze), 8vo, iv+216 pp., 1909; Volume II (Comunicazioni delle Sezioni I e II, matematica pura), 8vo, 315 pp., 1909; Volume III (Comunicazioni delle Sezioni III A, III B e IV, matematica applicata), 8vo, 583 pp., 1909.

AT a meeting of the Cambridge mathematical club on March 8 Sir ROBERT BALL delivered a lecture on Halley's comet.

The Club was founded four or five years ago with the hope of promoting the exchange and discussion of mathematical ideas among teachers of mathematics resident in Cambridge and the more proficient of their pupils. The meetings are held two or three times a term, and are usually comparatively small and quite informal, though occasionally a lecture or paper of a more formal character is presented. The subjects for discussion are sometimes branches of mathematical or physical theory, sometimes questions regarding mathematical teaching or examining. There is no publication in any way connected with the Club. The present President is Sir GEORGE DARWIN, and the Secretaries are Dr. T. J. I'A. BROMWICH and Mr. G. H. HARDY.

THE committee on the teaching of mathematics to engineering students, appointed at the Chicago joint meeting of 1907, will present its report at the meeting of the Society for the promotion of engineering education at Madison, Wis., June 23–25. Mathematicians who may be interested are invited to attend this important meeting.

THE mathematics section of the central association of science and mathematics teachers has issued a pamphlet entitled "Applied problems in algebra and geometry," compiled by a committee appointed for the purpose, containing 103 exercises taken from practical applications of these sciences. Copies may be secured, at 5 cents each, from the secretary of the section, Miss Mabel Sykes, Bowen High School, Chicago, Illinois.

TWO new chairs in mathematics have been established at the University of Paris, and one at Lyons and one at Toulouse; the occupants have not yet been announced.

THE following courses in mathematics are announced for the summer semester:

UNIVERSITY OF BERLIN. — By Professor H. A. SCHWARZ: Surfaces and space curves, four hours; Elliptic functions, four hours; Selected chapters in the theory of analytic functions, two hours; Colloquium, two hours; Seminar, two hours. — By Professor G. FROBENIUS: Theory of determinants, four hours; Seminar, two hours. — By Professor F. SCHOTTKY: Elementary theory of functions, four hours; Automorphic functions, four hours; Seminar, two hours. — By Professor R. LEHMANN-FILHÉS: Absolute perturbations according to Hansen, four hours; Analytic mechanics, four hours. — By Professor G.

HETTNER: Introduction to the theory of ordinary differential equations, two hours. — By Professor J. KNOBLAUCH: Applications of elliptic functions, four hours; Integral calculus, four hours; Selected chapters in differential calculus, four hours. — By Professor I. SCHUR: Differential calculus, with exercises, five hours; Theory of algebraic equations, two hours.

THE following courses in mathematics are announced for the academic year 1910–1911.

UNIVERSITY OF CHICAGO. — By Professor E. H. MOORE: Introduction to general analysis: Theory of functions of infinitely many variables; Integral equations in general analysis; Seminar on the foundations of pure mathematics; each two hours throughout the year. — By Professor L. E. DICKSON: Finite groups, four hours, first term; General algebra, four hours, second term; Quadratic forms, four hours, third term. — By Professor F. R. MOULTON: Modern theories of analytic differential equations with applications to celestial mechanics, four hours, all three terms. — By Professor E. J. WILCZYNSKI: Theory of plane curves, four hours, first term; Projective differential geometry of ruled surfaces and space curves, four hours, second term; Projective differential geometry of non-ruled surfaces and congruences, four hours, third term. — By Professor K. LAVES: Analytic mechanics, four hours, first and second terms. — By Professor H. E. SLAUGHT: Differential equations, four hours, first term. — By Professor G. A. BLISS: Elliptic integrals, four hours, second term; Theory of definite integrals, four hours, third term; Fundamental existence theorems, two hours, second and third terms. — By Dr. A. C. LUNN: Hydrodynamics, four hours, first term; Differential equations of mathematical physics, the conduction of heat, four hours, third term.

Summer Quarter, June 20 to September 2. — By Professor E. H. MOORE: General analysis, four hours; Seminar on the foundations of mathematics, four hours; Graphical methods in algebra, four hours, all second term. — By Professor L. E. DICKSON: Theory of substitutions, four hours; Differential calculus, five hours. — By Professor J. W. A. YOUNG: Critical review of secondary mathematics, four hours; Advanced algebra, five hours. — By Professor G. A. BLISS: Functions of a complex variable, four hours; Modern analytic geometry, four hours. — By Professor E. J. WILCZYNSKI: Projective differ-

ential geometry, four hours; Integral calculus, five hours; Synoptic course in mathematics, five hours. — By Professor A. L. UNDERHILL: Differential equations, five hours; Plane analytic geometry, five hours; College algebra, five hours.

COLUMBIA UNIVERSITY. — By Professor T. S. FISKE: Theory of functions of a real variable, three hours; Functions defined by linear differential equations, three hours. — By Professor F. N. COLE: Theory of functions of a complex variable, three hours; Theory of plane curves, three hours. — By Professor JAMES MACLAY: Differential equations, three hours, second half-year; Differential geometry, three hours, second half-year. — By Professor D. E. SMITH: History of mathematics, two hours; Seminar in the history and teaching of mathematics. — By Professor C. J. KEYSER: Modern theories in geometry, three hours; Principles of mathematics, three hours. — By Professor EDWARD KASNER: Vector analysis, two hours, first half-year; Geometry of differential equations, two hours.

Summer session (July 6 to August 17). — By Professor A. T. DELURY: Theory of functions of a complex variable, seven and one-half hours. — By Professor G. H. LING: Theory of groups of finite order, seven and one-half hours. — By Professor H. S. WHITE: Curves and surfaces of the third order, seven and one-half hours.

UNIVERSITY OF ILLINOIS. — By Professor S. W. SHATTUCK: Differential equations, three hours, first semester. — By Professor E. J. TOWNSEND: Theory of functions of a complex variable, three hours. — By Professor G. A. MILLER: Higher algebra, three hours, first semester; Theory of groups, three hours. — By Professor ———: Synoptic course, three hours; Differential geometry, three hours. — By Professor H. L. RIETZ: Actuarial theory, three hours, first semester; Theory of statistics, three hours. — By Professor J. W. ———: Elliptic functions, three hours. — By Professor C. H. SISAM: Algebraic surfaces, three hours. — By Dr. A. R. CRATHORNE: Advanced calculus, three hours, second semester; Theory of linear differential equations, three hours. — By Dr. R. L. BÖRGER: Projective geometry, three hours. — By Dr. G. E. WAHLIN: Partial differential equations, three hours, second semester. — By Dr. T. BUCK: Solid analytic geometry, three hours, second semester.

Summer of 1910. — By Professor G. A. MILLER: Theory of equations and determinants, five hours; Elementary theory of groups, three hours. — By Dr. E. B. LYTLE: Teachers' course,

five hours. — By Dr. G. E. WAHLIN: Differential equations, five hours.

INDIANA UNIVERSITY. — By Professor S. C. DAVISSON: Advanced calculus (a, w, s), three hours; Fourier series (a), three hours; Fundamental concepts of mathematics (w, s), two hours. — By Professor D. A. ROTHROCK: Systems of geometry (a, w), three hours; Calculus of variations (s, sm), three hours; History of mathematics (w), three hours. — By Professor U. S. HANNA: Theory of numbers (a), three hours; Substitution groups and Galois theory (w, s), three hours. — By Mr. K. P. WILLIAMS: Functions defined by differential equations (a, w), two hours.

(a, w, s, sm = autumn, winter, spring, summer.)

Summer Quarter, June 22–September 3, 1910. — By Professor S. C. DAVISSON: Advanced calculus, five hours; Modern analytic geometry, five hours. — By Professor D. A. ROTHROCK: History of mathematics, three hours; Ordinary differential equations, five hours. — By Professor U. S. HANNA: Advanced differential equations, five hours. — By Mr. K. P. WILLIAMS: Fourier series, three hours.

JOHNS HOPKINS UNIVERSITY. — By Professor F. MORLEY: Higher geometry, three hours, first half year; Theory of functions, three hours, second half year. — By Professor A. COHEN: Differential equations, two hours; Calculus of variations, two hours, first half year. — By Professor A. COBLE: Theory of groups, two hours; Theory of probabilities, two hours, second half year.

AT the meeting of the National academy of sciences held at Washington April 19–21, Professor F. R. MOULTON, of the University of Chicago, was elected to membership.

AT the meeting of the American philosophical society held on April 23 the following persons were elected to membership: President R. C. MACLAURIN, of the Massachusetts Institute of Technology, Professor B. O. PEIRCE, of Harvard University, Professor O. W. RICHARDSON, of Princeton University, and Professor C. E. PICARD, of the University of Paris.

PROFESSOR C. CARATHÉODORY, of the technical school at Hanover, has been appointed professor of mathematics at the newly established technical school at Breslau.

PROFESSOR R. DEDEKIND, of the technical school at Brunswick, has been elected foreign member of the academy of sciences at Paris.

PROFESSOR G. FABER, of the University of Tübingen, has been appointed professor of mathematics at the technical school of Stuttgart.

AT the University of Göttingen, Dr. A. HAAR and Dr. H. WEYL have been appointed docents in mathematics.

DR. W. SCHNEE has been appointed docent in mathematics at the University of Breslau.

MR. J. H. JEANS, formerly professor of mathematics in Princeton University, has been appointed Stokes lecturer in mathematics at Cambridge University.

AT Cornell University, Professors J. I. HUTCHINSON and VIGIL SNYDER have been promoted to full professorships of mathematics.

PROFESSOR OSKAR BOLZA, of the University of Chicago, has resigned and expects to reside in Freiburg, Germany. He will however retain his connection with the University as non-resident professor. Professor E. J. WILCZYNSKI, of the University of Illinois, has been appointed associate professor of mathematics at the University of Chicago.

AT the University of Wisconsin, Professor MAX MASON has been promoted to a full professorship of mathematical physics; Professor E. B. SKINNER has been promoted to an associate professor of mathematical physics; Mr. H. L. WOLF has been promoted to an assistant professorship of mathematics; Mr. S. E. URNER has been appointed instructor in mathematics.

PROFESSOR J. W. YOUNG, of the University of Illinois has been appointed head of the department of mathematics at the University of Kansas.

AT Columbia University Dr. C. C. GROVE has been made assistant professor of mathematics. Dr. N. J. LENNES has been appointed instructor in mathematics. Mr. C. B. UPTON has been promoted to an assistant professorship of mathematics in Teachers College. Professor JAMES MACLAY has been granted a half year's leave of absence which he will spend abroad.

AT the University of Pennsylvania hereafter the chairman of each department of instruction will be elected annually by

the department. For the year 1910–1911, Professor G. E. FISHER has been chosen chairman of the Department of Mathematics in the Graduate School, and Professor I. J. SCHWATT chairman of the Mathematical Department in the College.

AT the Georgia School of Technology, Mr. GEORGE RUTLEDGE and J. W. SPEAS have been appointed instructors in mathematics. Professor W. V. SKILES has been granted leave of absence to study at Harvard University. Mr. A. B. MORTON will resume his duties this fall, after a year's leave of absence at Brown University.

AT Stanford University, Mr. E. W. PONZER, of the University of Illinois, has been appointed assistant professor of mathematics; Mr. G. F. MCEWEN has been appointed instructor in applied mathematics.

MR. A. S. HAWKESWORTH has been appointed instructor in higher mathematics and lecturer in Semitic languages at the University of Pittsburgh.

AT the University of Minnesota Dr. H. L. SLOBIN has been made instructor in mathematics.

DR. ELIZABETH R. BENNETT has been appointed instructor in mathematics at the University of Nebraska.

RECENT catalogues of second hand mathematical works: Mayer & Müller, Prinz Louis Ferdinandstrasse 2, Berlin, N. W., catalogue No. 247, containing about 4,000 titles. — List & Francke, Talstrasse 2, Leipzig, catalogue No. 419, containing 1052 titles. Süddeutsches Antiquariat, Galleriestrasse 20, Munich, catalogue 23, containing 1123 titles.

NEW PUBLICATIONS.

(*In order to facilitate the early announcement of new mathematical books, publishers and authors are requested to send the requisite data as early as possible to the Departmental Editor,* PROFESSOR W. B. FORD, *1345 Wilmot Street, Ann Arbor, Mich.*)

I. HIGHER MATHEMATICS.

BERKELEY (H.). Mysticism in modern mathematic. Oxford University Press, 1910. 8vo. 12 + 264 pp. 8s.

BERNOULLI (J.). Ueber unendliche Reihen (1689–1704). Aus dem Lateinischen übersetzt und herausgegeben von G. Kowalewski. (Ostwald's Klassiker der exakten Wissenschaften.) Leipzig, Engelmann, 1909. 8vo. 141 pp. M. 2.50

BÖGER (R.). Projektive und analytische Schulgeometrie. Ein Lehr- und Uebungsbuch für die Oberklassen. Leipzig, Göschen, 1910. M. 3.60

BÜCHER, neue, über Naturwissenschaften und Mathematik. (Die Neuigkeiten des deutschen Buchhandels nach Wissenschaften geordnet.) Mitgeteilt Winter 1909–10. Leipzig, Hinrich. 8vo. Pp. 69–91. M. 0.30

COUTURAT (L.). Internaciona matematikal lexiko en Ido, Germana, Angla, Franca e Italiana. Internationales mathematisches Lexikon in Ido, Deutsch, Englisch, Französisch und Italienisch. Jena, Fischer, 1910. 8vo. 4 + 36 pp. M. 1.50

ENCYCLOPÉDIE des sciences mathématiques pures et appliquées. Tome II: Analyse. Vol. 3: Equations différentielles ordinaires. Fasc. 1: P. Painlevé, Existence de l'intégrale générale; détermination d'une intégrale particulière par ses valeurs initiales. E. Vessiot, Méthodes d'intégration élémentaires; étude des équations différentielles ordinaires au point de vue formel. Paris, Gauthier-Villars, 1910. 8vo. Pp. 1–170.

ESTANAVE (E.). Construction de modèles de surfaces applicables sur le paraboloide de révolution, surfaces définies de G. Darboux. (Mathematische Abhandlungen aus dem Verlage mathematischer Modelle von M. Schilling. Neue Folge. Nr. 7.) Leipzig, Schilling, 1909. 8vo. 22 pp. M. 1.20

GANTER (H.) und RUDIO (F.). Die Elemente der analytischen Geometrie. 1ter Teil. Die analytische Geometrie der Ebene. 7te, verbesserte Auflage. Leipzig, Teubner, 1910. M. 3.00

HARTENSTEIN (R.). Die Diskriminantenfläche der Gleichung 4ten Grades. (Mathematische Abhandlungen aus dem Verlage mathematischer Modelle von M. Schilling. Neue Folge. Nr. 8.) Leipzig, Schilling, 1909. 8vo. 19 pp. M. 1.20

JURETZKA (E.). Die Entwickelung unstetiger Funktionen nach den Eigenfunktionen des schwingenden Stabes auf Grund der Theorie der Integralgleichungen. (Diss.) Breslau, 1909. 8vo. 52 pp.

KOWALEWSKI (G.). See BERNOULLI (J.).

LIND (B.). Ueber das letzte Fermatsche Theorem. Leipzig, Teubner, 1910. 8vo. M. 2.00

PAINLEVÉ (P.). See ENCYCLOPÉDIE.

ROTHENBERG (S.). Geschichtliche Darstellung der Entwicklung der Theorie der singulären Lösungen totaler Differentialgleichungen von der ersten Ordnung mit 2 variablen Grössen. (Abhandlungen zur Geschichte der mathematischen Wissenschaften mit Einschluss ihrer Anwendungen, 20tes Heft, 3tes Stück.) Leipzig, Teubner, 1910. 8vo. Pp. 315–404. M. 3.60

RUDIO (F.). See GANTER (H.).

VESSIOT (E.). See ENCYCLOPÉDIE.

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